Buyer-Optimal Information with Nonlinear Technology: Supplemental Material

Kai Hao Yang*

July 09, 2019

S.1 Formal Definition of Virtual Valuation

The following constructions are results from Monteiro and Svaiter (2010). The purpose is to define formally the *virtual valuation*. Consider any CDF G with $\text{supp}(G) \subseteq [0,1]$. Let $a := \inf\{x \in [0,1] | G(x) > 0\}$ be the lower bound of $\text{supp}(G)$. Define

$$H(x|G) := \begin{cases} 0, & \text{if } x \in [0,a) \\ a - x(1-G(x)), & \text{if } x \in [a,1] \end{cases}.$$

Since G is increasing, $H(\cdot|G)$ is of bounded variation. Notice that for any measurable function $p : [0,1] \to [0,1]$,

$$\int_0^1 p(x) H(dx|G) = \int_0^1 p(x) x G(dx) - \int_0^1 p(x)(1-G(x))dx,$$

where the integral on the left hand side is defined with respect to the signed measure induced by $H(\cdot|G)$.

Let $\Theta := \{(\alpha, \beta) \in \mathbb{R}^2 | \alpha + \beta G(x) \leq H(x|G), \forall x \in [0,1]\}$ and let

$$\phi(x|G) := \sup\{\alpha + \beta G(x) | (\alpha, \beta) \in \Theta\}, \forall x \in [0,1]$$

be defined as the *generallized convex hull* of $H(\cdot|G)$ under measure G. We say that $w(x)$ is a *sub-gradient* of $\phi(\cdot|G)$ at $x \in [0,1]$ if

$$\phi(z|G) - \phi(x|G) \geq w(x)(G(z) - G(x)), \forall z \in [0,1].$$

*Department of Economics, University of Chicago, khyang@uchicago.edu.

For each $x \in [0,1]$, let $\partial\phi(x|G)$ denote the set of sub-gradients of $\phi(\cdot|G)$ at x. By Proposition 1(a) of Monteiro and Svaiter (2010), $\partial\phi(x|G) \neq \emptyset$ whenever $G(x) \in (0,1)$. Finally, let

$$\psi(x|G) := \begin{cases} \inf \partial\phi(x|G), & \text{if } \partial\phi(x|G) \neq \emptyset \\ -\infty, & \text{if } \partial\phi(x|G) = \emptyset \end{cases},$$

$\psi(\cdot|G) : [0,1] \to [-\infty, 1]$ is then defined as the *virtual valuation* induced by G.

S.2 Proof of Lemma 7

Proof of Lemma 7. Consider any $\nu \in \Delta(\mathcal{I}_\mu)$ such that $(\nu, \dots, \nu)$ is a Nash equilibrium and suppose that $\nu(\mathcal{R}\backslash\mathcal{I}_\mu^*) > 0$ for some $\mathcal{R} \not\subseteq \mathcal{I}_\mu^*$. Consider any $R \in \mathcal{R}\backslash\mathcal{I}_\mu^*$, let $r := \sup\{x \in [0,1] | \psi(x|R) < 0\}$ be the reserve price induced by R and let $a := \inf\{x \in [0,1] | R(x) > 0\}$, $b := \sup\{x \in [0,1] | R(x) < 1\}$ be the lower bound and upper of $\text{supp}(R)$, respectively. Since $R \in \mathcal{I}_\mu$,

$$1 - \mu - \int_0^1 R(x)\,\mathrm{d}x = 1 - \mu - \int_r^1 R(x)\,\mathrm{d}x - \int_a^r R(x)\,\mathrm{d}x = 0. \tag{S.1}$$

Also, since R is nondecreasing, it must be that

$$1 - \mu - \int_r^1 R(x)\,\mathrm{d}x - rR(r) \leq 1 - \mu - \int_r^1 R(x)\,\mathrm{d}x - (r-a)R(r) \leq 0.$$

In addition, for $(\nu, \dots, \nu)$ to be a symmetric equilibrium, it must be that γ_ν is continuous on $(r, 1)$, since otherwise, if there exists $q \in (0,1)$ such that $\gamma_\nu(q^-) < \gamma_\nu(q^+)$, then there exists $R \in \text{supp}(\nu)$ and an interval $(\underline{x}, \bar{x}) \subset (r, 1)$ such that $\psi(x|R) = q$ for all $x \in (\underline{x}, \bar{x})$. As such, for each buyer, there is a positive probability of tie when choosing R. One may then perturb R that breaks the tie under this event and wins the object with certainty at a small cost. Therefore, R cannot be a best response.

Now suppose that $\gamma_\nu(0) > 0$ and consider three cases separately:

Case 1: $R(r^-) > 0$ and

$$1 - \mu - \int_r^1 R(x)\,\mathrm{d}x - rR(r) < 0. \tag{S.2}$$

From Lemma 9, $(x - \psi(r^+|R))(1 - R(x)) \leq (r - \psi(r^+|R))(1 - R(r^-))$ for all $x \in [0,1]$. Therefore,

$$R(x) \geq \max\left\{0, 1 - \frac{\zeta}{(x - \psi(r^+|R))}\right\}. \tag{S.3}$$

for all $x \in [0, r]$, where $\zeta := (r - \psi(r^+|R))(1 - R(r^-))$. Moreover, since $\psi(x|R) < 0$ for all $x \in [0, r)$ and $R(r^-) > 0$ there exists $z \in [0, r)$ such that $R(x) > 1 - \zeta/(x - \psi(r^+|R)) > 0$

for all $x \in (z, r)$. Therefore,

$$1-\mu-\int_r^1 R(x)\,\mathrm{d}x-\int_0^r\left(1-\frac{\zeta}{(x-\psi(r^+|R))^+}\right)^+\mathrm{d}x>1-\mu-\int_0^1 R(x)\,\mathrm{d}x=0.$$

Together with (S.2), by the intermediate value theorem, there exists $\underline{x} \in (0, r)$ such that

$$1-\mu-\int_r^1 R(x)\,\mathrm{d}x-\int_{\underline{x}}^r\left(1-\frac{\zeta}{x-\psi(r^+|R)}\right)\mathrm{d}x-\underline{x}\left(1-\frac{\zeta}{\underline{x}-\psi(r^+|R)}\right)=0 \tag{S.4}$$

with $1-\zeta/(\underline{x}-\psi(r^+|R)) \geq 0$. Now define

$$\widehat{R}(x):=\begin{cases} R(x), & \text{if } x\in(r,1] \\ 1-\frac{\zeta}{x-\psi(r^+|R)}, & \text{if } x\in(\underline{x},r] \\ 1-\frac{\zeta}{\underline{x}-\psi(r^+|R)}, & \text{if } x\in[0,\underline{x}] \end{cases}.$$

By (S.4), $\widehat{R} \in \mathcal{I}_\mu$. Furthermore, by construction,

$$\psi(x|\widehat{R})=\begin{cases} \psi(x|R), & \text{if } x\in(r,1] \\ \psi(r^+|R), & \text{if } x\in(\underline{x},r] \\ \underline{\psi}, & \text{if } x\in[0,\underline{x}]. \end{cases},$$

for some $\underline{\psi} < 0$. Therefore,

$$\begin{aligned}
&\int_0^1 \gamma_\nu(\psi(x|\widehat{R}))(1-\widehat{R}(x))\,\mathrm{d}x \\
=&\int_{\underline{x}}^r \gamma_\nu(\psi(r^+|R))\left(1-\frac{\zeta}{x-\psi(r^+|R)}\right)\mathrm{d}x+\int_r^1 \gamma_\nu(\psi(x|R))(1-R(x))\,\mathrm{d}x \\
>&\int_r^1 \gamma_\nu(\psi(x|R))(1-R(x))\,\mathrm{d}x \\
=&\int_0^1 \gamma_\nu(\psi(x|R))(1-R(x))\,\mathrm{d}x,
\end{aligned}$$

where the strict inequality follows from $\gamma_\nu(0) > 0$ and $\psi(r^+|R) \geq 0$. Thus, there is some $\widehat{R} \in \mathcal{I}_\mu$ such that

$$\int_0^1 \gamma_\nu(\psi(x|R))(1-R(x))\,\mathrm{d}x<\int_0^1 \gamma_\nu(\psi(x|\widehat{R}))(1-\widehat{R}(x))\,\mathrm{d}x,$$

a contradiction. Therefore, $\mathcal{R}\backslash\mathcal{I}_\mu^*$ does not contain any R satisfying $R(r^-) > 0$ and (S.2) at the same time.

The conclusion in **case 1** leads to a further property about the set $\mathcal{R}\backslash\mathcal{I}_\mu^*$, as stated below.

Claim 4. For any $R \in \mathcal{R}\backslash\mathcal{I}^*_\mu$, there exists some $x_1, x_2 \in (r, 1]$ such that $\psi(x_2|R) > \psi(x_1|R) \geq 0$ and that $R(x_1) \leq R(x_2) < 1$.

Case 2:

$$1 - \mu - \int_r^1 R(x)\,\mathrm{d}x - rR(r) = 0. \tag{S.5}$$

Then by (S.1) and (S.5),

$$0 = 1 - \mu - \int_r^1 R(x)\,\mathrm{d}x - \int_a^r R(x) \geq 1 - \mu - \int_r^1 R(x)\,\mathrm{d}x - (r-a)R(r) = aR(r) \geq 0$$

and hence $R(r) = R(r^-)$, $\int_a^r (R(r) - R(x))\,\mathrm{d}x = 0$, $aR(r) = 0$ and therefore $R(r) = R(a)$.

By Claim 4, it must be that there exists some $x_0 \in (r, 1)$ and a sequence $\{x_n\}$ such that $x_n < x_{n+1} < x_0$ and $\psi(x_n|R) < \psi(x_{n+1}|R) < \psi(x_0^-|R)$ for all $n \in \mathbb{N}$ and that $\{x_n\} \uparrow x_0$. Since $\psi(x_{n+1}|R) > \psi(x_n|R)$ for all $n \in \mathbb{N}$ and $\psi(\cdot|R)$ is nondecreasing, we may take such $\{x_n\}$ so that $\psi(x_n^+|R) > \psi(y|R)$ for all $y \in [r, x_n)$ for all $n \in \mathbb{N}$. Let $k := \psi(x_0^-|R)$ and $\zeta_0 := (x_0 - k)(1 - R(x_0^-))$. Also, for each $n \in \mathbb{N}$, let $\zeta_n := (x_n - \psi(x_n^+|R))(1 - R(x_n^-))$. Then by Lemma 1, since $\psi(x_n^+|R) > \psi(y|R)$ for all $y \in [r, x_n)$ and since $\psi(x_{n+1}|R) > \psi(x_n|R)$, we must have

$$R(x) \geq \max\left\{1 - \frac{\zeta_0}{x-k},\, 1 - \frac{\zeta_n}{x - \psi(x_n|R)},\, 0\right\}, \tag{S.6}$$

for all $x \in [x_n, x_0]$, for all $n \in \mathbb{N}$, with strict inequality holding at some $x \in (x_n, x_0)$. Therefore, there exists $\bar{n} \in \mathbb{N}$ such that whenever $n > \bar{n}$, there exists $\underline{x}_n \in (r, x_n)$ and $\hat{x}_n \in (x_n, x_0)$ such that

$$\int_0^{x_0} R(x)\,\mathrm{d}x = \underline{x}_n R(\underline{x}_n) + \int_{\underline{x}_n}^{x_n} R(x)\,\mathrm{d}x + \int_{x_n}^{x_0} \max\left\{1 - \frac{\zeta_0}{x-k},\, 1 - \frac{\zeta_n}{x - \psi(x_n|R)}\right\}\mathrm{d}x \tag{S.7}$$

$$1 - \frac{\zeta_0}{\hat{x}_n - k} = 1 - \frac{\zeta_n}{\hat{x}_n - \psi(x_n|R)}. \tag{S.8}$$

As such, for any $n > \bar{n}$, define

$$\widehat{R}^{x_n}(x) := \begin{cases} R(x), & \text{if } x \in (x_0, 1] \\ 1 - \frac{\zeta_0}{x-k}, & \text{if } x \in (\hat{x}_n, x_0] \\ 1 - \frac{\zeta_n}{x - \psi(x_n|R)}, & \text{if } x \in (x_n, \hat{x}_n] \\ R(x), & \text{if } x \in (\underline{x}_n, \bar{x}_n] \\ R(\underline{x}_n), & \text{if } x \in [0, \underline{x}_n] \end{cases},$$

where $\hat{x}_n$ and $\underline{x}_n$ are uniquely defined by (S.7) and (S.8). Notice that by (S.6), $\hat{x}_n < \hat{x}_{n+1}$ for all $n \in \mathbb{N}$, $\{\hat{x}_n\} \uparrow x_0$ and $\underline{x}_n > \underline{x}_{n+1}$ for all $n \in \mathbb{N}$, $\{\underline{x}_n\} \downarrow r$.

By construction, for all $x \in [0,1]$, for any $n > \bar{n}$,

$$\psi(x|\widehat{R}^{x_n}) = \begin{cases} \psi(x|R), & \text{if } x \in (x_0, 1] \\ k, & \text{if } x \in (\hat{x}_n, x_0] \\ \psi(x_n|R), & \text{if } x \in (x_n, \hat{x}_n] \\ \psi(x|R), & \text{if } x \in (\underline{x}_n, x_n] \\ \underline{\psi}_n, & \text{if } x \in [0, \underline{x}_n] \end{cases},$$

for some $\underline{\psi}_n < 0$. As such, the difference in expected surplus between choosing $\widehat{R}^{x_n}$ and R is

$$\begin{aligned}
&\int_0^1 \gamma_\nu(\psi(x|\widehat{R}^{x_n}))(1-\widehat{R}^{x_n}(x))\,\mathrm{d}x - \int_0^1 \gamma_\nu(\psi(x|R))(1-R(x))\,\mathrm{d}x \\
=&\gamma_\nu(k)(r(1-R(r)) - \underline{x}_n(1-R(\underline{x}_n))) + \int_r^{x_n} (\gamma_\nu(k) - \gamma_\nu(\psi(x|R)))(1-R(x))\,\mathrm{d}x \\
&- \int_{x_n}^{\hat{x}_n} \big((\gamma_\nu(k) - \gamma_\nu(\psi(x_n|R)))(1-\widehat{R}^{x_n}(x))\big)\,\mathrm{d}x \\
&+ \int_{x_n}^{x_0} \big(\gamma_\nu(k) - \gamma_\nu(\psi(x|R)))(1-R(x))\big)\,\mathrm{d}x.
\end{aligned}$$

Notice that by Lemma 1 and $R(r) = R(r^-)$, $r(1-R(r)) = r(1-R(r^-)) \geq \underline{x}_n(1-R(\underline{x}_n))$ for all $n > \bar{n}$.

On the other hand, for any $\underline{x} \in (r, x_0)$, let

$$\begin{aligned}
\Psi(\underline{x}) :=& \int_r^{\underline{x}} \big(\gamma_\nu(k) - \gamma_\nu(\psi(x|R))\big)(1-R(x))\,\mathrm{d}x \\
&- \int_{\bar{x}(\underline{x})}^{\hat{x}(\underline{x})} \big(\gamma_\nu(k) - \gamma_\nu(\psi(\bar{x}(\underline{x})|R)\big)\big(1-\widehat{R}^{\underline{x}}(x)\big)\,\mathrm{d}x \\
&+ \int_{\bar{x}(\underline{x})}^{x_0} \big(\gamma_\nu(k) - \gamma_\nu(\psi(x|R))\big)(1-R(x))\,\mathrm{d}x,
\end{aligned} \tag{S.9}$$

where $\hat{x}(\underline{x})$ and $\bar{x}(\underline{x})$ are uniquely defined by

$$\int_0^{x_0} R(x)\,\mathrm{d}x = \underline{x}R(\underline{x}) + \int_{\underline{x}}^{\bar{x}(\underline{x})} R(x)\,\mathrm{d}x + \int_{\bar{x}(\underline{x})}^{x_0} \max\left\{1 - \frac{\zeta_0}{x-k},\, 1 - \frac{\zeta_{\bar{x}(\underline{x})}}{x - \psi(\bar{x}(\underline{x})|R)}\right\}\mathrm{d}x \tag{S.10}$$

$$1 - \frac{\zeta_0}{\hat{x}(\underline{x}) - k} = 1 - \frac{\zeta_{\bar{x}(\underline{x})}}{\hat{x}(\underline{x}) - \psi(\bar{x}(\underline{x})|R)}, \tag{S.11}$$

and $\widehat{R}^{\underline{x}}$ is defined by

$$\widehat{R}^{\underline{x}}(x) := \begin{cases} R(x), & \text{if } x \in (x_0, 1] \\ 1 - \frac{\zeta_0}{x-k}, & \text{if } x \in (\hat{x}(\underline{x}), x_0] \\ 1 - \frac{\zeta_{\bar{x}(\underline{x})}}{x - \psi(\bar{x}(\underline{x})|R)}, & \text{if } x \in (\bar{x}(\underline{x}), \hat{x}(\underline{x})] \\ R(x), & \text{if } x \in (\underline{x}, \bar{x}(\underline{x})] \\ R(\underline{x}), & \text{if } x \in [0, \underline{x}] \end{cases},$$

where $\zeta_x := (x - \psi(x|R))(1 - R(x^-))$ for any $x \in [r, 1]$.

Notice that by (S.7), (S.8), (S.10) and (S.11), for any $n > \bar{n}$, $x_n = \bar{x}(\underline{x}_n)$ and $\hat{x}_n = \hat{x}(\underline{x}_n)$. Also, $\hat{x}$ and $\bar{x}$ are in $\underline{x}$ and $\lim_{\underline{x}\to r} \bar{x}(\underline{x}) = \lim_{\underline{x}\to r} \hat{x}(\underline{x}) = x_0$, $\lim_{\underline{x}\to r} \psi(\bar{x}(\underline{x})|R) = \lim_{n\to\infty} \psi(x_n|R) = \psi(x_0^-|R)$.

Since $\psi(\cdot|R)$ and γ_ν are nondecreasing, by (S.10) and (S.11), $\bar{x}$ and $\hat{x}$ are differentiable Lebesgue-almost everywhere and therefore Ψ is differentiable Lebesgue-almost everywhere. Thus, for Lebesgue almost all $\underline{x}$,

$$\Psi'(\underline{x}) = (\gamma_\nu(k) - \gamma_\nu(\psi(\underline{x}|R)))(1 - R(\underline{x})) - \hat{x}'(\underline{x})(\gamma_\nu(k) - \gamma_\nu(\psi(\bar{x}(\underline{x})|R)))(1 - \widehat{R}^{\underline{x}}(\hat{x}(\underline{x}))) - \int_{\bar{x}(\underline{x})}^{\hat{x}(\underline{x})} \frac{\partial}{\partial \underline{x}}(\gamma_\nu(k) - \gamma_\nu(\psi(\bar{x}(\underline{x})|R)))(1 - \widehat{R}^{\underline{x}}(x))\,\mathrm{d}x.$$

Since $\lim_{\underline{x}\to r} \hat{x}(\underline{x}) = \lim_{\underline{x}\to r} \bar{x}(\underline{x}) = x_0$, $\lim_{\underline{x}\to r} \psi(\bar{x}(\underline{x})|R) = k$ and since $\hat{x}$ is decreasing, there exists $\delta > 0$ such that for $\underline{x}$ sufficiently close to r, whenever Ψ is differentiable at $\underline{x}$,

$$\Psi'(\underline{x}) > (\gamma_\nu(k) - \gamma_\nu(\psi(\underline{x}|R)))(1 - R(\underline{x})) - \delta > 0, \tag{S.12}$$

where the second inequality follows from $\gamma_\nu(k) - \gamma_\nu(\psi(r^+|R)) > 0$, which in turn follows from Claim 4 and that as $k > \psi(r^+|R)$. Therefore, since Ψ is continuous in $\underline{x}$ and $\Psi'(\underline{x}) > 0$ whenever Ψ is differentiable at $\underline{x}$ and $\underline{x}$ is sufficiently close to r, there exists $\hat{r}$ such that $\Psi(\underline{x}) > 0$ for all $\underline{x} \in (r, \hat{r})$.

As such, for n sufficiently large so that $\underline{x}_n \in (r, \hat{r})$,

$$\begin{aligned} 0 < \Psi(\underline{x}_n) = &\int_r^{\underline{x}_n} (\gamma_\nu(k) - \gamma_\nu(\psi(x|R)))(1 - R(x))\,\mathrm{d}x \\ &- \int_{x_n}^{\hat{x}_n} \big((\gamma_\nu(k) - \gamma_\nu(\psi(x_n|R)))(1 - \widehat{R}^{x_n}(x))\big)\,\mathrm{d}x \\ &+ \int_{x_n}^{x_0} \big(\gamma_\nu(k) - \gamma_\nu(\psi(x|R)))(1 - R(x))\big)\,\mathrm{d}x. \end{aligned}$$

Together, for n sufficiently large,

$$\int_0^1 \gamma_\nu(\psi(x|\widehat{R}^{x_n}))(1 - \widehat{R}^{x_n}(x))\,\mathrm{d}x > \int_0^1 \gamma_\nu(\psi(x|R))(1 - R(x))\,\mathrm{d}x,$$

a contradiction.

If, on the other hand, for any $x \in (r, 1)$ and for any sequence $\{x_n\}$ such that $\{x_n\} \uparrow x$, there exists $\bar{n} \in \mathbb{N}$ such that $\psi(x_n|R) = \psi(x^-|R)$ for all $n > \bar{n}$, then for any $x \in (r, 1)$, there exists $\delta > 0$ such that $\psi(y|R) = \psi(x^-|R)$ for all $y \in (x - \delta, x)$. Let $\delta_x := \sup\{\delta > 0 | \psi(y|R) = \psi(x^-|R), \forall y \in (x - \delta, x)\}$. Then $\delta_x > 0$ for all $x \in (r, 1)$. Moreover, for any $x, y \in (r, 1)$, if $\psi(x^-|R) \neq \psi(y^-|R)$, then it must be that $(x - \delta_x, x) \cap (y - \delta_y, y) = \emptyset$. Therefore, $\{\psi(x^-|R)\}_{x\in(r,1)}$ is at most countable. Since $\psi(\cdot|R)^+$ is nondecreasing, it must be a step function. Now let $b := \sup\{x \in [r,1] | R(x) < 1\}$. Consider first the case when for any $\delta > 0$, there exists $x, y \in (b-\delta, b)$ with $x < y$ such that $\psi(x|R) < \psi(y|R) < \psi(b^-|R)$. Since $\psi(\cdot|R)^+$ is a step function, $\psi(\cdot|R)^+$ has countably infinitely many jumps and therefore we may represent $\psi(\cdot|R)^+$ as

$$\psi(x|R) = \sum_{n=1}^{\infty} \psi_n \mathbf{1}\{x \in (\alpha_n, \beta_n)\}, \, \forall x \in [r,1] \backslash [\{\alpha_n\}_{n=1}^{\infty} \cup \{\beta_n\}_{n=1}^{\infty}].$$

for some $\{\alpha\}_n$, $\{\beta_n\}$ with $\beta_n > \alpha_n$ for all $n \in \mathbb{N}$ and some $\{\psi_n\}_{n=1}^{\infty}$ such that $\psi_n > \psi(r^+|R)$ for all $n \in \mathbb{N}$, $n \geq 2$. Since for any $\delta > 0$, there exists $x, y \in (b-\delta, b)$, $x < y$, such that $\psi(x|R) < \psi(y|R) < \psi(b^-|R)$, there exists a sequence $\{x_j\}$ such that $x_j \in \{\alpha_n\}_{n=1}^{\infty} \cup \{\beta_n\}_{n=1}^{\infty}$, $x_j < x_{j+1}$, $\psi_j := \psi(x_j^+|R) = \psi(x_{j+1}^-|R) < \psi(x_{j+1}^+|R) =: \psi_{j+1}$ for all $j \in \mathbb{N}$, and that $\{x_j\} \uparrow b$ and $\{\psi_j\} \uparrow \psi(b^-|R)$.

Since $\psi(\cdot|R)^+$ is a step function and $\psi(x|R) = b$ for all $x \in [b,1]$, we must have $\psi(b^-|R) < \psi(b|R) = b$ and $R(b^-) < 1$. As such, let $\zeta_b := (b - \psi(b^-|R))(1 - R(b^-))$. Then $\zeta_b > 0$. Also, for each $j \in \mathbb{N}$, let $\zeta_j := (x_j - \psi_j)(1 - R(x_j^-))$, by Lemma 1, since $\psi_{j-1} < \psi_j < \psi_{j+1}$, we have

$$R(x) > \max\left\{1 - \frac{\zeta_b}{x - \psi(b^-|R)}, 1 - \frac{\zeta_j}{x - \psi_j}\right\},$$

for all $x \in (x_j, b)$.

As such, there exists a sequence $\{r_j\}$ such that $\{r_j\} \downarrow r$ and $\widehat{R}^j \in \mathcal{I}_\mu$ for j large enough, where

$$\widehat{R}^j(x) := \begin{cases} R(x), & \text{if } x \in (b,1] \\ 1 - \frac{\zeta_b}{x - \psi(b^-|R)}, & \text{if } x \in (\hat{x}_j, b] \\ 1 - \frac{\zeta_j}{x - \psi_j}, & \text{if } x \in (\bar{x}_j, \hat{x}_j] \\ R(x), & \text{if } x \in (r_j, \bar{x}_j] \\ R(r_j), & \text{if } x \in [0, r_j] \end{cases}.$$

and $x_j < \hat{x}_j < \bar{x}_j < b$ are uniquely defined by

$$\int_0^b \widehat{R}^j(x)\,\mathrm{d}x = \int_0^b R(x)\,\mathrm{d}x$$
$$1 - \frac{\zeta_b}{\hat{x}_j - \psi(b^-|R)} = 1 - \frac{\zeta_j}{\hat{x}_j - \psi_j},$$

Similar to the previous case, for each $j \in \mathbb{N}$ such that $\widehat{R}^j \in \mathcal{I}_\mu$, the deviation gain from

R to $\widehat{R}^j$ is

$$\gamma_\nu(\psi(b^-|R))(r(1-R(r))-r_j(1-R(r_j)))+\int_r^{r_j}\big(\gamma_\nu(\psi(b^-|R))-\gamma_\nu(\psi(x|R))\big)(1-R(x))\,\mathrm{d}x$$
$$-\int_{\bar{x}_j}^{\hat{x}_j}\big(\gamma_\nu(\psi(b^-|R))-\gamma_\nu(\psi_j)\big)(1-\widehat{R}^j(x))\,\mathrm{d}x$$
$$+\int_{\bar{x}_j}^{x_j}\big(\gamma_\nu(\psi(b^-|R))-\gamma_\nu(\psi(x|R))\big)(1-R(x))\,\mathrm{d}x.$$

As noted above, from Lemma 1, $r(1-R(r))=r(1-R(r^-))\geq r_j(1-R(r_j))$ for all $j\in\mathbb{N}$. Also, as shown in the previous case, from (S.9) and (S.12), with $x_0=b$ and $k=\psi(b^-|R)$, there exists $\delta>0$ such that for $\underline{x}$ sufficiently close to r, $\Psi'(\underline{x})>\gamma_\nu(\psi(b^-|R))-\gamma_\nu(\psi(r^+|R))-\delta>0$ by Claim 4 and $\psi_j>\psi(r^+|R)$ for all $j\in\mathbb{N}$. Thus, since by (S.10) and (S.11), $\hat{x}_j=\hat{x}(r_j)$ and $\bar{x}(r_j)=\bar{x}_j$, and since $\{r_j\}\downarrow r$, for j sufficiently large,

$$\begin{aligned}0<\Psi(r_j)=&\int_r^{r_j}\big(\gamma_\nu(\psi(b^-|R))-\gamma_\nu(\psi(x|R))\big)(1-R(x))\,\mathrm{d}x\\&-\int_{\bar{x}_j}^{\hat{x}_j}\big(\gamma_\nu(\psi(b^-|R))-\gamma_\nu(\psi_j)\big)(1-\widehat{R}^j(x))\,\mathrm{d}x\\&+\int_{\bar{x}_j}^{x_j}\big(\gamma_\nu(\psi(b^-|R))-\gamma_\nu(\psi(x|R))\big)(1-R(x))\,\mathrm{d}x.\end{aligned}$$

Together, for j large enough, $\widehat{R}^j\in\mathcal{I}_\mu$ and

$$\int_0^1\gamma_\nu(\psi(x|\widehat{R}^j))(1-\widehat{R}^j(x))\,\mathrm{d}x>\int_0^1\gamma_\nu(\psi(x|R))(1-R(x))\,\mathrm{d}x,$$

a contradiction.

Finally, if there exists $\delta>0$ such that for any $x,y\in(b-\delta,b)$, $\psi(x|R)=\psi(y|R)=\psi(b^-|R)$. Let $\underline{b}:=\inf\{b_0\in[r,1)|\psi(x|R)=\psi(y|R),\forall x,y\in(b_0,b)\}$ and let $\underline{\psi}:=\psi(b^-|R)$. Then $\underline{b}<b$ and $\underline{\psi}>\psi(r^+|R)$ since $\psi(\cdot|R)$ is not a constant on $[r,b)$. We claim that it is without loss to suppose that $\phi(x|R)=H(x|R)$ for all $x\in(\underline{b},b)$. Indeed, if there exists $x\in(\underline{b},b)$ such that $\phi(x|R)<H(x|R)$, let $\zeta:=(\underline{b}-\underline{\psi})(1-R(\underline{b}^-))$. Lemma 1 then ensures that

$$R(x)\geq 1-\frac{\zeta}{x-\underline{\psi}},$$

for all $x\in(\underline{b},b)$ with strict inequality for some $x\in(\underline{b},b)$ and therefore, there exists $\hat{b}\in(\underline{b},b)$ such that

$$\int_{\underline{b}}^{b}(1-R(x))\,\mathrm{d}x=\int_{\underline{b}}^{\hat{b}}\frac{\zeta}{x-\underline{\psi}}\,\mathrm{d}x.$$

Therefore, for

$$\widehat{R}(x):=\begin{cases}R(x), & \text{if } x\in[0,\underline{b})\\ 1-\frac{\zeta}{x-\underline{\psi}}, & \text{if } x\in[\underline{b},\hat{b})\\ 1, & \text{if } x\in[\hat{b},1]\end{cases},$$

$\widehat{R} \in \mathcal{I}_\mu$ and

$$\int_0^1 \gamma_\nu(\psi(x|R))(1-R(x))\,\mathrm{d}x = \int_0^1 \gamma_\nu(\psi(x|\widehat{R}))(1-\widehat{R}(x))\,\mathrm{d}x$$

and $H(x|\widehat{R}) = \phi(x|\widehat{R})$ for all $x \in (\underline{b}, b)$.

Therefore, as $H(x|R) = \phi(x|R)$ for all $x \in (\underline{b}, b)$ and $\psi(x|R) = \underline{\psi}$ for all $x \in (\underline{b}, b)$, we must have

$$R(x) = 1 - \frac{\zeta}{x - \underline{\psi}},\ \forall x \in [\underline{b}, b),$$

for some $\zeta > 0$. Now take and fix any $\bar{x} \in (\underline{b}, b)$ and notice that for any $h \in (\psi(\underline{b}^-|R), \underline{\psi})$ and any $k \in (\underline{\psi}, b)$, let $\zeta(k) := (\bar{x} - k)(1 - R(\bar{x}^-))$ and $\zeta(h) := (\underline{b} - h)(1 - R(\underline{b}^-))$, we must have

$$R(x) > \max\left\{1 - \frac{\zeta(k)}{x-k}, 1 - \frac{\zeta(h)}{x-h}\right\},$$

for all $x \in (\underline{b}, \bar{x})$. Thus, for any $\underline{x} > r$ that is close enough to r, there exists $h(\underline{x}) \in (\psi(\underline{b}^-|R), \underline{\psi})$,$k(\underline{x}) \in (\underline{\psi}, b)$ and $\hat{x}(\underline{x}) \in (\underline{b}, \bar{x})$ such that $\lim_{\underline{x}\downarrow r} h(\underline{x}) = \lim_{\underline{x}\downarrow r} k(\underline{x}) = \underline{\psi}$,

$$\frac{\zeta(k(\underline{x}))}{\hat{x}(\underline{x}) - k(\underline{x})} = \frac{\zeta(h(\underline{x}))}{\hat{x}(\underline{x}) - h(\underline{x})}$$

and

$$\int_0^{\bar{x}} R(x)\,\mathrm{d}x = \int_0^{\bar{x}} \widehat{R}^{\underline{x}}(x)\,\mathrm{d}x,$$

where

$$\widehat{R}^{\underline{x}}(x) := \begin{cases} R(x), & \text{if } x \in [\bar{x}, 1] \\ 1 - \frac{\zeta(k(\underline{x}))}{x - k(\underline{x})}, & \text{if } x \in [\hat{x}(\underline{x}), \bar{x}) \\ 1 - \frac{\zeta(h(\underline{x}))}{x - h(\underline{x})}, & \text{if } x \in [\underline{b}, \hat{x}(\underline{x})) \\ R(x), & \text{if } x \in [\underline{x}, \underline{b}) \\ R(\underline{x}), & \text{if } x \in [0, \underline{x}] \end{cases}$$

and thus $\widehat{R}^{\underline{x}} \in \mathcal{I}_\mu$. Moreover, such $h(\cdot)$ and $k(\cdot)$ can be selected so that $\hat{x}(\underline{x})$ is decreasing in $\underline{x}$ and $\lim_{\underline{x}\downarrow r} \hat{x}(\underline{x}) = \bar{x}$.

Notice that for any such $\widehat{R}^{\underline{x}}$,

$$\psi(x|\widehat{R}^{\underline{x}}) = \begin{cases} \underline{\psi}, & \text{if } x \in [0, \underline{x}] \\ \psi(x|R), & \text{if } x \in (\underline{x}, \underline{b}] \\ h(\underline{x}), & \text{if } x \in (\underline{b}, \hat{x}(\underline{x})] \\ \tilde{k}(\underline{x}), & \text{if } x \in (\bar{x}(\underline{x}), b) \\ b, & \text{if } x \in [b, 1]. \end{cases},$$

for some $\underline{\psi} < 0$ and some $\tilde{k}(\underline{x}) \in (\underline{\psi}, k(\underline{x}))$. As such, for any $\underline{x}$ such that $\widehat{R}^{\underline{x}} \in \mathcal{I}_\mu$, the deviation gain from R to $\widehat{R}^{\underline{x}}$ is

$$\begin{aligned}
&\int_0^1 \gamma_\nu(\psi(x|\widehat{R}^{\underline{x}}))(1-\widehat{R}^{\underline{x}}(x))\,\mathrm{d}x - \int_0^1 \gamma_\nu(\psi(x|R))(1-R(x))\,\mathrm{d}x \\
&> \gamma_\nu(\tilde{k}(\underline{x}))(r(1-R(r)) - \underline{x}(1-R(\underline{x}))) + \int_r^{\underline{x}} (\gamma_\nu(\tilde{k}(\underline{x})) - \gamma_\nu(\psi(x|R)))(1-R(x))\,\mathrm{d}x \\
&- \int_{\underline{b}}^{\hat{x}(\underline{x})} (\gamma_\nu(\tilde{k}(\underline{x})) - \gamma_\nu(h(\underline{x})))(1-\widehat{R}^{\underline{x}}(x))\,\mathrm{d}x,
\end{aligned}$$

where the strict inequality follows from $\tilde{k}(\underline{x}) > \underline{\psi}$ and Claim 4.

Again, by Lemma 1, $r(1-R(r)) \geq \underline{x}(1-R(\underline{x}))$ for all $\underline{x} \geq r$. Also, let

$$\Psi(\underline{x}) := \int_r^{\underline{x}} (\gamma_\nu(\tilde{k}(\underline{x})) - \gamma_\nu(\psi(x|R)))(1-R(x))\,\mathrm{d}x - \int_{\underline{b}}^{\hat{x}(\underline{x})} (\gamma_\nu(\tilde{k}(\underline{x})) - \gamma_\nu(h(\underline{x})))(1-\widehat{R}^{\underline{x}}(x))\,\mathrm{d}x.$$

Then Ψ is differentiable Lebesgue-almost everywhere and the derivative converges to

$$(\gamma_\nu(\psi(\underline{b}^-|R)) - \gamma_\nu(\psi(r^+|R)))(1-R(r)) - \lim_{k,h\to\underline{\psi},\, k>h} (\gamma_\nu(k)-\gamma_\nu(h)) \cdot \lim_{\underline{x}\downarrow r} \hat{x}'(\underline{x}) \cdot (1-R(\bar{x})) \geq 0$$

as $\underline{x}$ approaches r, which follows from Claim 4 and the properties that $\hat{x}(\underline{x})$ is decreasing in $\underline{x}$ and $\psi(b^-|R) > \psi(r^+|R)$.

Together, for $\underline{x}$ close enough to r, there exists $\widehat{R}^{\underline{x}} \in \mathcal{I}_\mu$ such that

$$\int_0^1 \gamma_\nu(\psi(x|\widehat{R}^{\underline{x}}))(1-\widehat{R}^{\underline{x}}(x))\,\mathrm{d}x > \int_0^1 \gamma_\nu(\psi(x|R))(1-R(x))\,\mathrm{d}x,$$

a contradiction.

Case 3: $R(r^-) = 0$

Notice that the arguments in **case 2** rely only on the observations that $R(r) = R(r^-) = R(a)$ and $aR(r) = 0$. As such, if $R(r) = R(r^-) = 0$, this is exactly **case 2**. On the other hand, if $R(r) > R(r^-) = 0$, we may define

$$G(x) := \begin{cases} R(r), & \text{if } x \in [0, r) \\ R(x), & \text{if } x \in [r, 1]. \end{cases}.$$

Then since the the arguments in **case 2** do not depend on particular value of $\mu = \int_0^1 (1 - R(x))\,\mathrm{d}x$, by the same arguments, there exists $\widehat{G} \in \mathcal{I}_{\bar{\mu}}$ such that

$$\int_0^1 \gamma_\nu(\psi(x|\widehat{G}))(1-\widehat{G}(x))\,\mathrm{d}x > \int_0^1 \gamma_\nu(\psi(x|G))(1-G(x))\,\mathrm{d}x,$$

and that $G(\underline{x}) = \widehat{G}(\underline{x}) = \widehat{G}(0)$ for some $\underline{x} \in (r, 1)$, where $\bar{\mu} := \int_0^1 (1 - G(x))\,\mathrm{d}x$. Since $\underline{x} > r$ and $G(\underline{x}) \geq G(r) = R(r)$, there exists $\epsilon > 0$ such that $\underline{x}(\widehat{G}(\underline{x}) - \epsilon) = rR(r)$ and therefore $\widehat{R} \in \mathcal{I}_\mu$, where

$$\widehat{R}(x) := \begin{cases} \widehat{G}(\underline{x}) - \epsilon, & \text{if } x \in [0, \underline{x}) \\ \widehat{G}(x), & \text{if } x \in [\underline{x}, 1] \end{cases}.$$

Furthermore, by construction, $\psi(x|\widehat{G}) = \psi(x|\widehat{R})$ for all $x \in (\underline{x}, 1]$, $\psi(x|G) = \psi(x|R)$ for all $x \in (r, 1]$, $\psi(x|\widehat{G}) < 0$ if and only if $\psi(x|\widehat{R}) < 0$ and $\psi(x|G) < 0$ if and only if $\psi(x|R) < 0$. Together, we have

$$\begin{aligned} \int_0^1 \gamma_\nu(\psi(x|R))(1 - R(x))\,\mathrm{d}x &= \int_0^1 \gamma_\nu(\psi(x|G))(1 - G(x))\,\mathrm{d}x \\ &< \int_0^1 \gamma_\nu(\psi(x|\widehat{G}))(1 - \widehat{G}(x))\,\mathrm{d}x \\ &= \int_0^1 \gamma_\nu(\psi(x|\widehat{R}))(1 - \widehat{R}(x))\,\mathrm{d}x \end{aligned}$$

for some $\widehat{R} \in \mathcal{I}_\mu$, a contradiction.

Finally, if $\gamma_\nu(0) = 0$, then it must be that for any $R \in \mathrm{supp}(\nu)$,

$$r_R := \sup\{x \in [0, 1] | \psi(x|R) < 0\} = \inf\{x \in [0, 1] | R(x) > 0\} := a_R.$$

and therefore $R(r_R^-) = 0$ for any $R \in \mathrm{supp}(\nu)$. Notice that the arguments in **case 2** and **case 3** does not rely on assumption that $\gamma_\nu(0) > 0$. Thus, if there exists $\mathcal{R} \nsubseteq \mathcal{I}_\mu^*$ such that $\nu(\mathcal{R} \backslash \mathcal{I}_\mu^*) > 0$ for any $R \in \mathcal{R} \backslash \mathcal{I}_\mu^*$, the previous arguments are still valid since we will never have the case when $\psi(\cdot|R)$ is not a constant and **case 1** holds. As a result, there exists $\widehat{R} \in \mathcal{I}_\mu$ such that

$$\int_0^1 \gamma_\nu(\psi(x|R))(1 - R(x))\,\mathrm{d}x < \int_0^1 \gamma_\nu(\psi(x|\widehat{R}))(1 - \widehat{R}(x))\,\mathrm{d}x,$$

a contradiction.

Together, there does not exist any $\mathcal{R} \subseteq \mathcal{I}_\mu$ such that $\nu(\mathcal{R} \backslash \mathcal{I}_\mu^*) > 0$. This implies that $\mathrm{supp}(\nu) \subseteq \mathcal{I}_\mu^*$. ■

S.3 Proof of Claims

Proof of Claim 1. We first show that there are at most countably many $x \in \mathrm{supp(G)}$ such that $\partial\phi(x|G)$ is not a singleton. Indeed, consider any $x, y \in [0, 1]$ such that $G(y) > G(x)$. Take any $\beta_x \in \partial\phi(x|G)$, $\beta_y \in \phi(y|G)$. By definition

$$\phi(y|G) - \phi(x|G) \geq \beta_x(G(y) - G(x))$$

and

$$\phi(x|G) - \phi(y|G) \geq \beta_y(G(x) - G(y)).$$

Adding up, we have

$$(\beta_y - \beta_x)(G(y) - G(x)) \geq 0$$

and therefore $\beta_y \geq \beta_x$. As such, the family of intervals

$$\{(\inf \partial\phi(x|G), \sup \partial\phi(x|G))|x \in \mathrm{supp}(G)\}$$

is pairwise disjoint and thus there is at most countably many $x \in \mathrm{supp}(G)$ such that $\inf \partial\phi(x|G) < \sup \partial\phi(x|G)$, as desired.

Moreover, by Lemma 3 in Monteiro (2015), $\bar{\psi}(x) \in \partial\phi(x|G)$. Therefore, for any $x \in$ supp(G) such that $\partial\phi(x|G)$ is a singleton, $\bar{\psi}(x) = \hat{\psi}(x|G)$. Together, there exists a countable set $\mathbb{C} \subseteq$ supp(G) such that $\bar{\psi}(x) = \hat{\psi}(x|G)$ for all $x \in$ supp(G)$\backslash\mathbb{C}$. This completes the proof. ■

Proof of Claim 2. Let $\{x_k\}_{k=1}^K$ be the collection of jumps. Since $\{x_k\}_{k=1}^K$ is finite, there exists $\bar{\delta}$ such that the intervals $\{[x_k - \delta, x_k + \delta]\}_{k=1}^K$ are disjoint for all $\delta \in (0, \bar{\delta})$. For each $m \in \mathbb{N}$ such that $1/m \in (0, \bar{\delta})$, define G_m as

$$G_m(x) := \begin{cases} G(x), & \text{if } x \notin \cup_{k=1}^n [\underline{x}_k^m, \bar{x}_k^m] \\ G(\underline{x}_k^m) + \alpha_k \left(\frac{1}{\underline{x}_k^m - \psi(\underline{x}_k^m|G)} - \frac{1}{x - \psi(\underline{x}_k^m|G)} \right), & \text{if } x \in [\underline{x}_k^m, x_k],\ k \in \{1, \dots, K\} \\ G(\bar{x}_k^m) - \alpha_k \left(\frac{1}{\bar{x}_k^m - \psi(\bar{x}_k^m|G)} - \frac{1}{x - \psi(\bar{x}_k^m|G)} \right), & \text{if } x \in (x_k, \bar{x}_k^m],\ k \in \{1, \dots, K\} \end{cases},$$

where

$$\alpha_k := \frac{G(\bar{x}_k^m) - G(\underline{x}_k^m)}{\frac{1}{\underline{x}_k^m - \psi(\underline{x}_k^m|G)} - \frac{1}{\bar{x}_k - \psi(\bar{x}_k^m|G)}}; \quad \underline{x}_k^m := x_k - \frac{1}{m}; \quad \bar{x}_k^m := x_k + \frac{1}{m}.$$

Then $\{G_m\} \to G$ under the weak-* topology and G_m is continuous for all $m \in \mathbb{N}$. Furthermore, $\{\psi(\cdot|G_m)\} \to \psi(\cdot|G)$ pointwisely Lebesgue-almost everywhere and therefore, as γ is right-continuous on $[0, 1]$,

$$\limsup_{n\to\infty} \gamma(\psi(x|G_m)) \leq \gamma(\psi(x|G))$$

for Lebesgue-almost all $x \in [0, 1]$. Together, by the Reverse Fatou's Lemma, since $\gamma \leq 1$,

$$\limsup_{m\to\infty} \int_0^1 \gamma(\psi(x|G_m))(1 - G_m(x))\,\mathrm{d}x \leq \int_0^1 \gamma(\psi(x|G))(1 - G(x))\,\mathrm{d}x,$$

as desired. ■

Proof of Claim 3. Let $\mu_G \in \Delta([0,1])$ denote the probability measure associated with G, ν_G denote the signed measure associated with $H(\cdot|G)$ and let D denote the set of jumps of G. For any $m \in \mathbb{N}$, since $\mu_G([0,1]) = 1 < \infty$ and $|\nu_G|([0,1]) = 1 < \infty$, there exists a finite subset $D_m \subset D$ such that $\mu_G(D \backslash D_m) < 1/4m$ and $|\nu_G|(D\backslash D_m) < 1/4m$. For each $m \in \mathbb{N}$, define a probability measure μ_m by moving all the mass points on $D\backslash D_m$ uniformly to D_m. That is, $\mu_m(A) := \mu_G(A)$ for any $A \in \mathcal{B}([0,1])$ such that $A \cap D = \emptyset$, $\mu_m(\{d\}) := 0$ for all $d \in D\backslash D_m$ and $\mu_m(\{d\}) := 1/4m|D_m|$. Let G_m be the CDF associated with the measure μ_m. It then follows that $\{G_m\} \to G$ under the weak-* topology and G_m has finitely many jumps for all $m \in \mathbb{N}$.

Let $\sigma(\psi(\cdot|G))$ and $\sigma(\psi(\cdot|G_m))$ be the sigma-algebras generated by $\psi(\cdot|G)$ and $\psi(\cdot|G_m)$, respectively. Notice that $\sigma(\psi(\cdot|G)) \subseteq \sigma(\psi(\cdot|G_m))$ for all $m \in \mathbb{N}$. Therefore, by Theorem 3 in Monteiro and Svaiter (2010), for any $\lambda : [0,1] \to [0,1]$ that is $\sigma(\psi(\cdot|G))$-measurable

$$\begin{aligned}
\frac{1}{2m} >& \left| \sum_{d \in D_m} \lambda(d)\nu(\{d\}) - \sum_{d \in D_m} \lambda(d) \left(\nu(\{d\} - \frac{1}{4m|D_m|}) \right) + \sum_{d \in D \backslash D_m} \lambda(d)\nu(\{d\}) \right| \\
\geq& \left| \int_0^1 \lambda(x) H(\mathrm{d}x|G) - \int_0^1 \lambda(x) H(\mathrm{d}x|G_m) \right| \\
=& \left| \int_0^1 \lambda(x)\psi(x|G)G(\mathrm{d}x) - \int_0^1 \lambda(x)\psi(x|G_m)G_m(\mathrm{d}x) \right| \\
=& \left| \int_{D\backslash D_m} \lambda(x)\psi(x|G)\mu_G(\mathrm{d}x) - \int_{D\backslash D_m} \lambda(x)\psi(x|G_m)\mu_m(\mathrm{d}x) + \int_{D_m} \lambda(x)\psi(x|G)\mu_G(\mathrm{d}x) \right. \\
&\left. - \int_{D_m} \lambda(x)\psi(x|G_m)\mu_m(\mathrm{d}x) + \int_{[0,1]\backslash D} \lambda(x)(\psi(x|G) - \psi(x|G_m))\mu_G(\mathrm{d}x) \right| \\
>& \left| \sum_{d \in D_m} \lambda(d)\psi(d|G)\mu_G(\{d\}) - \sum_{d \in D_m} \lambda(d)\psi(x|G_m) \Bigg(\mu_G(\{d\}) \right. \\
&\left. + \frac{1}{4m|D_m|} \Bigg) + \int_{[0,1]\backslash D} \lambda(x)(\psi(x|G) - \psi(x|G_m))\mu_G(\mathrm{d}x) \right| - \left| \int_{D\backslash D_m} \lambda(x)\psi(x|G)\mu_G(\mathrm{d}x) \right| \\
\geq& \left| \int_0^1 \lambda(x)(\psi(x|G) - \psi(x|G_m))\mu_G(\mathrm{d}x) \right| - \frac{1}{2m}
\end{aligned}$$

and thus

$$\left| \int_0^1 \lambda(x)(\psi(x|G) - \psi(x|G_m))\mu_G(\mathrm{d}x) \right| < \frac{1}{m}.$$

In particular, since $\gamma \circ \psi(\cdot|G) : [0,1] \to [0,1]$ is $\sigma(\psi(\cdot|G))$-measurable,

$$\left| \int_0^1 \gamma(\psi(x|G))(\psi(x|G) - \psi(x|G_m))G(\mathrm{d}x) \right| < \frac{1}{m} \tag{S.13}$$

and therefore

$$\lim_{m\to\infty} \left| \int_0^1 \gamma(\psi(x|G))(\psi(x|G) - \psi(x|G_m))G(\mathrm{d}x) \right| = 0. \tag{S.14}$$

Together with Lipschitz continuity of γ on $(0,1)$, which implies that $|\gamma(y)-\gamma(x)| \leq K|y-x|$ for some $K>0$, for all $x,y \in (0,1)$, we have

$$\begin{aligned}
&\lim_{m\to\infty}\left|\int_0^1 \gamma(\psi(x|G))(x-\psi(x|G))G(\mathrm{d}x) - \int_0^1 \gamma(\psi(x|G_m))(x-\psi(x|G_m))G_m(\mathrm{d}x)\right| \\
\leq &\lim_{m\to\infty}\Bigg[\left|\int_0^1 \gamma(\psi(x|G))(\psi(x|G)^+ - \psi(x|G_m)^+)G(\mathrm{d}x)\right| \\
&+\left|\int_0^1 (x-\psi(x|G_m)^+)(\gamma(\psi(x|G)) - \gamma(\psi(x|G_m)))G(\mathrm{d}x)\right| \\
&+\left|\int_0^1 \gamma(\psi(x|G_m))(x-\psi(x|G_m)^+)G(\mathrm{d}x) - \int_0^1 \gamma(\psi(x|G_m))(x-\psi(x|G_m)^+)G_m(\mathrm{d}x)\right|\Bigg] \\
\leq &0 + \lim_{m\to\infty}\left|\int_0^1 (x-\psi(x|G_m)^+)K(\psi(x|G)^+ - \psi(x|G_m)^+)G(\mathrm{d}x)\right| + \lim_{m\to\infty}\frac{1}{2m} \\
\leq &2K\cdot\lim_{m\to\infty}\left|\int_0^1 (\psi(x|G)^+ - \psi(x|G_m)^+)G(\mathrm{d}x)\right| \\
= &0,
\end{aligned}$$

where the second inequality follows from the Lipschitz property of γ and (S.14) and the third inequality follows from (S.13), the fact that $\sup_{x\in[0,1]}|x-\psi(x|G)^+| \leq 2$, that a constant function 1 is $\sigma(\psi(\cdot|G))$-measurable and from the definition of μ_m.

As such,

$$\begin{aligned}
&\lim_{m\to\infty}\int_0^1 \gamma(\psi(x|G_m))(1-G_m(x))\,\mathrm{d}x = \lim_{m\to\infty}\int_0^1 \gamma(\psi(x|G_m))(x-\psi(x|G_m))G_m(\mathrm{d}x) \\
=&\int_0^1 \gamma(\psi(x|G))(x-\psi(x|G))G(\mathrm{d}x) = \int_0^1 \gamma(\psi(x|G))(1-G(x))\,\mathrm{d}x.
\end{aligned}$$

In particular,

$$\limsup_{n\to\infty}\int_0^1 \gamma(\psi(x|G_m))(1-G_m(x))\,\mathrm{d}x \leq \int_0^1 \gamma(\psi(x|G))(1-G(x))\,\mathrm{d}x,$$

as desired. ∎

Proof of Claim 4. First notice that if $\psi(x|R)=\bar{\psi}$ for all $x\in[r,b]$ for some $\bar{\psi}$, since $R \notin \mathcal{I}^*_\mu$, we must have $\phi(x|R) < H(x|R)$ for some $x \in (r,b)$. By Lemma 1, let $\zeta := (r-\bar{\psi})(1-R(r^-))$, then

$$R(x) \geq 1 - \frac{\zeta}{x-\bar{\psi}}$$

for all $x\in[0,1]$. Since $\phi(x|R) < H(x|R)$ for some $x\in(r,b)$, the inequality is strict for some $x\in(r,b)$ such that $1-\zeta/(x-\bar{\psi}) > 0$. By **case 1** of the proof of Lemma 2 (which does not rely on Claim 4), it must be that either $R(r^-)=0$ or

$$1-\mu-\int_r^1 R(x)\,\mathrm{d}x - rR(r) = 0,$$

which further implies that $R(r) = R(r^-) = R(a)$ and $aR(r) = 0$, as shown in the proof of Lemma 1 and the proof of Lemma 2.

Therefore, there exists $\bar{b} \in [\mu, b)$ such that for

$$\widehat{R}(x) := \begin{cases} 0, & \text{if } x \in \left[0, \frac{\zeta}{1-R(r^-)} + \bar{\psi}\right) \\ 1 - \frac{\zeta}{x-\bar{\psi}}, & \text{if } x \in \left[\frac{\zeta}{1-R(r^-)} + \bar{\psi}, \bar{b}\right) \\ 1, & \text{if } x \in [\bar{b}, 1]. \end{cases},$$

$\widehat{R} \in \mathcal{I}_\mu$. Then,

$$\int_0^1 \gamma_\nu(\psi(x|R))(1 - R(x))\,\mathrm{d}x = \gamma_\nu(\bar{\psi}) \int_r^1 (1 - R(x))\,\mathrm{d}x = \gamma_\nu(\bar{\psi}) \left(\mu - r(1 - R(r^-))\right)$$

$$\leq \gamma_\nu(\bar{\psi})(\mu - \zeta - \bar{\psi}(1 - R(r^-))) = \int_0^1 \gamma_\nu(\psi(x|\widehat{R}))(1 - \widehat{R}(x))\,\mathrm{d}x.$$

Furthermore, since $\bar{b} < b < 1$, there exists $\pi \in [0, 1 - \bar{\psi}]$, $\eta \in [0, 1 - \mu]$ such that

$$\int_0^1 \gamma_\nu \left(\psi \left(x \middle| G^\eta_{\pi,\bar{\psi}}\right)\right) \left(1 - G^\eta_{\pi,\bar{\psi}}(x)\right) \mathrm{d}x = \gamma_\nu(\bar{\psi})(\mu - \pi - (1-\eta)\bar{\psi}) > \gamma(\bar{\psi})(\mu - \zeta - \bar{\psi}(1 - R(r^-))).$$

Together, R cannot be a best response. Since $\nu(\mathcal{R} \backslash \mathcal{I}^*_\mu) > 0$, it must be that for any $R \in \mathcal{R} \backslash \mathcal{I}^*_\mu$, there exists $x_1, x_2 \in [r, b]$ such that $\psi(x_2|R) > \psi(x_1|R)$, as desired. ■

References

Monteiro, P. K. (2015): "A Note on the Continuity of the Optimal Auction," *Economics Letters*, 137, 127–130.

Monteiro, P. K. and B. F. Svaiter (2010): "Optimal Auction with a General Distribution: Virtual Valuation without Densities," *Journal of Mathematical Economics*, 46, 21–31.