Information, Bargaining Power and Efficiency: Re-examining the Role of Incomplete Information in Crisis Bargaining.

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Abstract

In this paper, without fully specifying the underlying game form, we showed that the probability of an inefficient breakdown in any bilateral crisis bargaining model is smaller when the more informed party has more bargaining power. Moreover, introduction of additional private information does not necessarily lead to extra efficiency loss. Several implications can be drawn from these results. Specifically, regarding international security, reducing incomplete information is not the only way to reduce the probability of war. Instead, reallocating bargaining power properly would also be effective in terms of preventing conflicts. Furthermore, these results also provide a formal justification for the power transition theory as the *status-quo* power can be interpreted as the party with more bargaining power when the information structure shifts due to power transition.

Keywords: Mechanism design, bargaining with incomplete information, crisis bargaining, informed principal, international conflict.

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1 Introduction

Bargaining between two parties to divide a fixed amount of surplus is an ubiquitous scenario in various economic and political realms. For instance, bargaining between a firm and a union over some wage package; bargaining between a rebel group and the incumbent government over some political resources; or bargaining between two sovereign countries. A salient common nature of these bargaining scenarios is that bargaining breakdowns often lead to an inefficient outcome and thus it is of common interest for both parties to achieve an agreement instead of failing the bargaining processes. However, despite the fact that breakdowns are (ex-post) inefficient, such outcomes nevertheless occur frequently (e.g. strikes, rebellion or war). Particularly, in the study of international conflicts, there is a long-existing puzzle why countries often fail to reach agreements before wars even though it is common knowledge for both sides that war is costly and thus inefficient. What prevents countries, as rational actors, from achieving an outcome that is mutually beneficial for them in a crisis bargaining? In a vast scale of literature, such inefficiencies are attributed to the presence of *incomplete information* and the *incentives to misrepresent private information.* It is argued that the privately known information, together with the incentives to misrepresent it, impose additional constraints on the set of feasible allocations and thus crisis bargaining, being an interim stage, cannot effectively achieve the *ex-post* efficient allocation, even though it is common knowledge to both parties. However, despite its scale, few of these studies has provided a systematic description about *how* incomplete information would affect efficiency and *how* are information and efficiency related in a general crisis bargaining framework.

The studies that focus on crisis bargaining and inefficient bargaining breakdowns can be categorized into two classes. The first, also referred as the *rationalist explanation of war* literature, follows Fearon's (1995) seminal paper.[1] Studies in this class consider various forms of game theoretic models and draw implications from the equilibria about inefficient breakdowns. Specifically, Fearon (1995) considered a take-it-or-leave-it bargaining game between two countries with one of the countries having private information on cost of war (resolution) and showed that, even though it is commonly known that war is ex-post inefficient, private information still leads to a positive

[1] In fact, Fearon (1995) posited three main explanations for wars between rational states. The other two causes are *commitment problem* and *indivisibility.* Although there are also following studies on these two causes, we will focus on the incomplete information explanation in this paper. Such selection is not implying that incomplete information is the *only* cause of inefficient breakdowns. Instead, it aims to provide further and more systematic insights on the role of incomplete information in affecting efficiency of crisis bargaining.

probability of war. On the other hand, Reed (2003) adopted the same game form but instead supposed that the private information is on the distribution of power. It is then shown that probability of war is also positive. Moreover, some connections between the dispersion of information and the distribution of power are identified and are used to draw implications about the relationship between stability and distribution of power. In addition, Powell (1996a, 1996b) considered a modified alternate bargaining model á la Rubinstein (1982). Under an infinite-horizon, alternate offering game in which costs of war are private information and options of using force to terminate bargaining are allowed whenever players are making decisions, the (unique perfect Bayesian) equilibrium outcome when satisfied and dissatisfied state makes the first offer, respectively, are characterized and used to draw implications about relationships among status quo, distribution of power and probability of breakdowns. As a result, it is shown that probability of war is zero if and only if the discrepancy between status quo and distribution of power is small enough. In some following papers, Powell (2004) further allowed one side to make consecutive offers in a similar alternate offering model but with battles in between. Slantchev (2003), in the same way, allowed both offer and counter offers with battles in between.[2] Furthermore, Wittman (2009) constructed a (simultaneous offering) model in the spirit of Chatterjee & Samuelson (1983), in which demand and offer are made at the same time and uncertainty is on the expected outcome of war. The (non-trivial) equilibrium then gives positive probability of war, even when both sides are jointly pessimistic, but the winning probabilities are different from the alternating offering models. Beside these studies, various other forms of models are also considered in this class of literature. (see also: Filson & Werner, 2003; Leventoglu & Tahar, 2008; Schultz, 1999; Wagner, 2000.)

The other class of studies (often referred as *model-free* approach), contrary to the first class, does not fully specify any game forms for the bargaining process and bargaining protocol. Instead, they use the techniques in mechanism design and the revelation principle to characterize the necessary and sufficient conditions for inefficient breakdowns that apply to *any* equilibrium in *any* bargaining protocol and thus are robust to model details. Specifically, Banks (1990) considered an environment with one-sided incomplete information on cost of war. Fey & Ramsey (2011) extended the above results to various information structures, including two sided uncertainty on costs and outcomes

[2] As one may notice here, in this category of literature, war might not always be modeled as the end of the bargaining process. Instead, battles are also possible during the process in some models. However, among all these models that permit battles during the process, there are still some terminations that correspond to wars in models without battles. As it will be clearer later in this paper, at our level of generality, these distinctions do not matter and thus we will reduce all the interpretations of "war" as the failure of bargaining *at the end of bargaining process*.

of war, with independent types, provided further necessary and sufficient conditions for existence of always peaceful protocols and some properties of probabilities of war and equilibrium payoffs in any protocols. In another companion paper, Fey & Ramsey (2008) also showed similar results with correlated types which follow a common uniform prior.

Although studies in the first category—through a complete specification of the underlying protocol *a priori* and the characterizations of equilibria in these models—are able to provide detailed descriptions of how the information structure affect probabilities of inefficient breakdowns and how are they related, such descriptions are not systematic as they depend heavily on the specifications of the underlying protocols and other model details. The results and implications may be significantly different under different specifications of game forms and model-dependent assumptions, yielding considerable difficulties for generally understanding the way through which information affects efficiency. For instance, although in the alternate-offering model of Powell (1996a), probability of war is positive in equilibrium, Leventoglu & Tahar (2008) showed that there is an equilibrium that admits zero probability of war if the player who makes the first offer is altered. On the other hand, despite the similarity of the information structure between Wittman's (2009) simultaneous offering model and Reed's (2003) take-it-or-leave-it model, the outcomes are quite different. Such sensitivity, as remarked by Fey & Ramsey (2011), is particularly problematic for the study of international conflict as a whole discipline since unlike domestic politics or other economic environments, there is often no obvious bargaining protocols that exist in the first place.

On the other hand, although the generality and detail-free feature of the second category allow these studies to circumvent the above problems and provide results and implications that are robust to all possible specifications of bargaining protocols and model details, such generalities also prevent them from deriving sharper explanations and predictions about how incomplete information is related to efficiency in general. All the results in this category are either conditions under which it is *possible* for there to exist an always peaceful equilibrium or general relationships between private information and the (interim) probability of inefficient breakdowns (e.g. monotonicity), yet there are no further implications about which protocol will prevail as an outcome nor descriptions about how efficiencies would eventually perform given the preliminaries of the environment and hence, as a result, are lacking for further understanding of the role of information in crisis bargaining.

This paper takes an intermediate approach on the spectrum between the two categories described above. It provides further descriptions of the role of incomplete information and how it is related to efficiency without fully specifying model details. As a result, it may integrate the sparse and potentially contradicting results in the first category into a systematic description regarding the relationship between information structure and efficiency that also reflects the generality and model-free feature of the second category and provide further extensions to the results there. In brief, the main results in this paper suggest that probabilities of inefficient breakdowns do not depend on the presence of incomplete information *alone*, but is in fact determined by how well the allocation of *bargaining power* is aligned with the underlying information structure. Specifically, the main results indicate that, the efficiency loss caused by presence of incomplete information is less when the underlying information structure is better aligned with the allocation of bargaining powers. This implication then enables us to understand how the information structure and efficiency are related in a systematic way.

To state these results more formally, we first need a further understanding of *bargaining power*. However, in the leading literature of crisis bargaining, there is no formal and systematic definition of bargaining powers. Indeed, among these studies, the concept of bargaining power is often defined ambiguously and is intertwined with various model details. Among various models in the first category, bargaining power is related to the timing of the model, the order of making offers and counteroffers, the availability of options to opt-out or terminate the process unilaterally. All the features above are sensitive to *a priori* specifications of model details, which is also the reason why the second approach often has few words on the bargaining power. Moreover, particularly in the studies of international conflicts, the *anarchic* structure in the international realm creates an environment in which no obvious bargaining protocols are in their existence at the first place. Due to the lack of central authority, unlike bargaining in some economic realms, there would be no credible force to design and to impose bargaining protocols *prior* to the beginning of any crisis bargaining. The above two observations therefore suggest that the concept of bargaining power, in order to be defined systematically and to be consistent with the substantive nature of international relations, should be considered without specific bargaining protocols.

To this end, we adopt Myerson's (1983) interpretation of (full) bargaining power:

> "*If an individual is the principal, it means that he has effective control over the channels of communications between all individuals [...]. If the principal designs a game for the subordinates to play, then he can be confident that they will use the strategies which he suggests for them in this game, provided that these strategies form a Nash equilibrium.*"

Formally, we define a party with *full bargaining power* as the one who has the right to select the underlying bargaining protocol (game form) for a crisis bargaining scenario (henceforth, this party will be called principal and the other will be called subordinate). As remarked above, although the principal has the right of selection, it does not mean that she can obtain whatever outcome she would like to. Instead, there are certain constrains on the *feasible* protocols that can be chosen and successfully implemented. The first constraint is as noted in the above quote. Given any selected protocol and suppose that both parties agree to bargain according to this protocol. The *outcomes* that can be expected from this protocol should be the ones in an *equilibrium* in the game induced by the protocol. By the revelation principle, this is equivalent to saying that a *feasible* protocols for the principal must be *incentive compatible*. Second, the anarchy nature of international relation plays another rule in crisis bargaining, namely the lack of central authority enables any party to unilaterally terminate (i.e. initiate a war) the bargaining process whenever they prefer to. As such, any feasible protocol for the principal must exhibit this feature, which is often referred as *voluntary agreement* in the literature (Fey & Ramsey, 2011). That is, all the feasible protocol must also incorporate an option for each party to terminate the bargaining and hence leads to breakdown unilaterally. In other words, a feasible protocol must also be *individually rational*. These constraints sometimes, however, impose some difficulties to the principal more than themselves. Since the principal would also be a player in such protocol once it is proposed, when the principal possesses private information, there would be an additional trade-off to the principal. On one hand, the principal would tend to select the protocol that is the best for her "type". On the other hand, such selection would then reveal some private information of the principal, leading to a change of incentive compatibility and individual rationality constraints and reduce the potential gains for the principal during the actual bargaining. Therefore, by imposing the aforementioned definition of bargaining powers, the problem would essentially become a *mechanism design problem of an informed principle.* (Myerson, 1983, 1984; Maskin & Tirole, 1990, 1992)

By adopting this definition of bargaining power, we are then able to endogenize the emergence of bargaining protocol without fully specifying them *a priori*. Methodologically, this integrates the two categories in the leading literature into an intermediate one, in which bargaining protocols are not fully specified but some specific yet systematic descriptions about the relationship between information structure and efficiency can still be derived. Substantively, this interpretation is seemingly to better capture the feature of crisis bargaining in international relations. The two main features resulting from the anarchic nature—That there is no central authority that is capable of enforcing effectively and credibly and that both parties can unilaterally initiate a war whenever they prefer to—are both embedded in this framework.

There are several papers that are closer to this paper. Substantively, this paper is closer to Leventoglu (2012) and Hörner et al. (2015). Leventoglu (2012) also considers a bargaining problem between two countries where bargaining protocols are themselves objects of the bargaining and thus is also an endogenous mechanism design problem with a potentially informed principal. While she uses a two-types example, focuses on the possibility of breakdowns and adopt the approach from Maskin & Tirole (1990, 1992), this paper considers an environment with a continuum of types, focus also on how the probability of breakdowns differs when alignment between bargaining power and information structure changes, and adopt Myerson's (1983) approach. Hörner et al. (2015) also deal with an optimal mechanism design problem in order to minimize the probability of war. However, their main focus is on an environment where there is a (neutral) mediator between two parties, which is different from that of this paper. Methodologically, this paper is similar to Fey & Ramsey (2008; 2011) in that it also takes a mechanism design approach and applies the revelation principle to study all the possible game forms without fully specifying them. However, unlike Fey & Ramsey, the mechanisms in this paper are designed endogenously whereas Fey & Ramsey consider all the possible mechanisms but do not pin down the mechanism that eventually emerges.

The rest of this paper will be organized as follows. In section 2 we will introduce all the preliminaries. Section 3 and 4 will include our main result, with full bargaining power on one side and uncertainties on private value and common value respectively. Section 5 extends the results to accommodate intermediate bargaining powers. Section 6 discusses the theoretical results and their implications. Section 7 then concludes. Before the next section, we first introduce a motivating example which exhibits many of the insights in the main results.

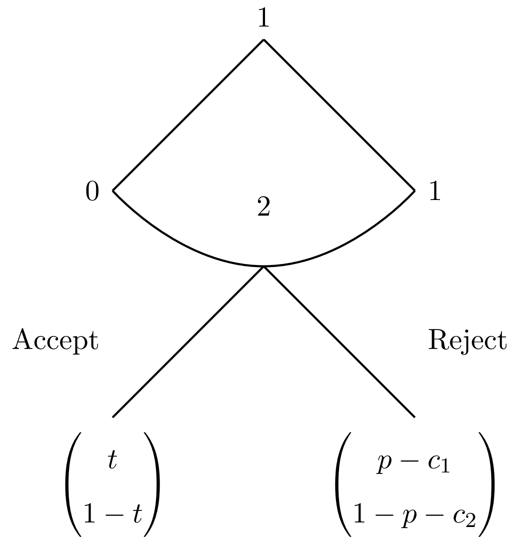

Figure 1.

Consider the take-it-and-leave-it model in Fearon (1995). Formally, suppose that there are two countries, 1 and 2. Country 1 offers a share $1-t \in [0,1]$ of an divisible good whose size is normalized to 1. Country 2 then decides whether to accept this offer or not. If 2 accepts, then surplus will be divided according to $(t, 1-t)$ whereas if 2 rejects, a war occurs. Whenever a war occurs, both countries' payoff are determined by their *relative power* and *costs/resolutions*. Let $p \in [0,1]$ denote the probability that 1 wins if a war occurs, in which case 1 gets all the surplus. Such p then captures the relative power between two countries. Also let $c_1 \in [\underline{c_1}.\overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$ denote the costs of war for 1 and 2 respectively. Therefore, payoffs when a war occurs are $(p - c_1, 1 - p - c_2)$. Assume that $\underline{c_1} > 0$, $\underline{c_2} > 0$ so that war is always inefficient and that $\overline{c_1} < \infty, \overline{c_2} < \infty$. The game form is described in Figure 1.

Consider first the case that country 2 has private information about c_2, which is exactly the model in Fearon (1995). Formally, suppose that $c_2 \sim F_2$, where F_2 admits strictly positive density f_2 on $[\underline{c_2}, \overline{c_2}]$ and has increasing hazard rate. i.e. the function $c_2 \mapsto \frac{f_2(c_2)}{1-F_2(c_2)}$ is increasing. Suppose also that only 2 can observe the realization of c_2. Then as shown in Fearon (1995), in the unique (up to a tie breaking rule) perfect Bayesian equilibrium in this game, 1 offers t^* and 2 rejects if and only if $1 - t \geq 1 - p - c_2$ for any offer t, where t^* is given by:

$$t^* - p + c_1 = \frac{1 - F_2(t^* - p)}{f_2(t^* - p)},$$

if the solution exists in $[p + \underline{c_2}, p + \overline{c_2}]$ and $t^* = p + \underline{c_2}$ if not. Notice that in this equilibrium, if

$t^* > p + \underline{c}_2$, (ex-ante) probability of war is positive.[3]

Now consider instead the case when 1 has private information. Namely $c_1 \sim F_1$ for some CDF F_1 and only 1 can observe the realization of c_1 while c_2 is common knowledge. Then in any perfect Bayesian equilibrium, 1 offers $t^* = p + c_2$, 2 rejects if and only if $1 - t < 1 - p - c_2$ for any offered t. In particular, probability of war is zero in any perfect Bayesian equilibrium.

Finally, consider the case when both sides have private information. That is, $c_1 \sim F_1$ and $c_2 \sim F_2$, where F_2 has strictly positive density f_2 and increasing hazard rate and only 1 can observe realization of c_1, 2 can observe realization of c_2. Then in any perfect Bayesian equilibrium, 1 offers t^* and 2 rejects if and only if $1 - t < 1 - p - c_2$ for any t being offered, where t^* is defined as in the first case. In this case, the probability of war, given any realization of c_1, is the same as if c_1 were common knowledge, which is characterized in the first case.

Together, even under the same game form, the above three different information structures could possibly yield different outcomes. As only player 1 can make offers, it is natural to say that 1 has full bargaining power under this particular game form. There are two notable features in the above example. First, in the first case where 1 has full bargaining power but incomplete information, the probability of war is positive, whereas in the second case, where 1 has full bargaining power and full information, probability of war is zero. Intuitively, when 1 has full information, together with her full bargaining power, she can precisely extract all the surplus from 2. Since war is inefficient, 1 would avoid it to maximize total surplus. However, when 1 does not have full information, she cannot retain such preciseness anymore. When 1 faces uncertainties on 2's resolution, the *risk-return trade-off* then enters in effect. 1 now has to "guess" 2's resolution level and choose a proper offer that balances the potential gain from making higher offer and the potential risk that the offer might be rejected. If the distribution is not too "condensed" on the low costs, the full support assumption then ensures that there is positive probability that the costs for 2 would be low enough so that 1's offer would be unacceptable for such realizations. The potentially "wrong guess" of the one making offer is then the driving force of positive probability of breakdown. Second, even though the first case exhibits positive probability of war, comparing that to the third case, we can also see that, when extra private information is introduced to the party with full bargaining power, there would be no additional efficiency loss. The idea is similar, since extra private information to the party with full bargaining power does not exacerbate the risk-return trade-off problem.

From the above rather simple example, we can see that there are indeed some relationships

[3] This is often called this *risk-return trade-off* in the literature.

among information, bargaining power and efficiency. The rest of this paper will be dedicated to understanding these features in a more general setting, in particular, the settings that are robust to model details.

2 Preliminaries

In this section, we will introduce the main preliminaries for the analyses. Specifically, we will introduce the environment and preference structures, define the bargaining protocols, formulate the constraints and solution concepts. To begin with, we will consider the payoff structures as they are in the crisis bargaining literature. As in the previous example, suppose that there is a fixed (normalized to be of size 1), divisible surplus to be divided by two players, 1 and 2. If $p \in [0,1]$ denotes the probability of 1 winning a war, c_1, c_2 denote costs of war to 1 and 2 respectively, $t \in [0,1]$ denote the share of surplus that 1 will obtain had there been no war and $w \in [0,1]$ be the probability of war. Then (expected) payoffs for 1 and 2 are respectively:

$$(1-w)t + w(p - c_1)$$
$$(1-w)(1-t) + w(1-p-c_2).$$

On the other hand, as in the mechanism design literature, a bargaining protocol is defined by a collection of strategy spaces[4], together with an outcome function that maps strategy profiles to possible outcomes which satisfies the *voluntary agreement* assumption. Formally,

Definition 1. A bargaining protocol is defined by a tuple $(\mathbb{A}_1, \mathbb{A}_2, w, t)$ such that:

1. $\mathbb{A}_1 = (A_1, \mathcal{A}_1)$, $\mathbb{A}_2 = (A_2, \mathcal{A}_2)$ are measurable spaces with sigma algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ (strategy space),
2. $(w,t) : A_1 \times A_2 \to [0,1]^2$ and is $\mathcal{A}_1 \times \mathcal{A}_2$-measurable (outcome functions).
3. $w(a_1^0, \cdot) \equiv w(\cdot, a_2^0) \equiv 1$ for some $a_1^0 \in A_1$, $a_2^0 \in A_2$ (voluntary agreement).

In words, a bargaining protocol specifies the strategies available for all players and determines the outcomes, which is consisting of the probability of breakdowns and the rule of surplus division if the bargaining does not breakdown given any strategies that are chosen by the players.

[4]Strictly speaking, this is a slight abuse of terminology. Although it will be seen later that we will be considering Bayesian games after specifying the underlying information structures, the term *strategies* here are not referring to the strategies in such induced Bayesian game—measurable functions that maps types to actions. Instead these strategies are the strategies in a fictitious game induced by the *game form* as if there were no private information.

In the following analyses, we will first suppose that some party has full bargaining power and thus the right to select the bargaining protocols and consider various environments under different information structures. Specifically, we will consider *private value* environments, in which one's private information only (directly) affects his/her payoff (i.e. uncertainties on c_1 and c_2) as well as *common value* environments, where one's private information may affect both parties' payoff (directly) (i.e. uncertainties on p). These would then allow us to compare how efficiency evolves as the relationship between the underlying information structure and allocation of bargaining power shifts. We first define the setting in a more general way. Suppose that player 1 and 2 can observe private signals θ_1, θ_2, respectively, where $\theta_i \in \Theta_i$ and $\theta_i \sim G_i$, $\Theta_i \subseteq \mathbb{R}$ is a compact interval and G_i is a CDF for each $i \in \{1,2\}$. We assume throughout the paper that θ_1 and θ_2 are independent. Payoffs when bargaining breaks down are then determined by the realizations of θ_1, θ_2 and functions $p : \Theta_1 \times \Theta_2 \to [0,1]$, $\{c_i : \Theta_i \to [\underline{c_i}, \overline{c_i}]\}_{i=1}^2$, where $\underline{c_i} > 0$[5] for each $i \in \{1,2\}$. With this, a Bayesian Nash equilibrium in the game induced by a protocol is a pair of $(\mathcal{A}_i)$ measurable functions (s_1^*, s_2^*), $s_i^* : \Theta_i \to A_i$ such that

$$
\begin{aligned}
&\mathbb{E}_{G_2}[(1 - w(s_1^*(\theta_1), s_2^*(\theta_2)))t(s_1^*(\theta_1), s_2^*(\theta_2)) + w(s_1^*(\theta_1), s_2^*(\theta_2)))(p(\theta_1,\theta_2) - c_1(\theta_1))] \\
\geq &\mathbb{E}_{G_2}[(1 - w(a_1, s_2^*(\theta_2)))t(a_1, s_2^*(\theta_2)) + w(a_1, s_2^*(\theta_2)))(p(\theta_1,\theta_2) - c_1(\theta_1))], \forall a_1 \in A_1, \\
&\mathbb{E}_{G_1}[(1 - w(s_1^*(\theta_1), s_2^*(\theta_2)))(1 - t(s_1^*(\theta_1), s_2^*(\theta_2))) + w(s_1^*(\theta_1, s_2^*(\theta_2)))(1 - p(\theta_1,\theta_2) - c_2(\theta_2))] \\
\geq &\mathbb{E}_{G_1}[(1 - w(s_1^*(\theta_1), a_2)(1 - t(s_1^*(\theta_1), a_2)) + w(s_1^*(\theta_1), a_2)(1 - p(\theta_1,\theta_2) - c_2(\theta_2))], \forall a_2 \in A_2.
\end{aligned}
$$

By the revelation principle, for any bargaining protocol and any Bayesian Nash equilibrium in the induced game, there is an incentive compatible direct protocol such that outcomes are equivalent. Therefore, we can without loss of generality restrict the set of protocols for the principal to be the set of incentive compatible direct protocols that satisfy the counterpart of voluntary agreement assumption. Henceforth, we will simply refer a direct protocol $m = (w,t)$ as a bargaining protocol, where $(w,t) : \Theta_1 \times \Theta_2 \to [0,1]^2$. Thus, given a protocol $m = (w,t)$, (ex-post) payoffs for 1 and 2, when reporting θ_1', θ_2' under true signals θ_1, θ_2, are respectively:

$$
\begin{aligned}
&(1 - w(\theta_1', \theta_2'))t(\theta_1', \theta_2') + w(\theta_1', \theta_2')(p(\theta_1,\theta_2) - c_1(\theta_1)) \\
&(1 - w(\theta_1', \theta_2'))(1 - t(\theta_1', \theta_2')) + w(\theta_1', \theta_2')(1 - p(\theta_1,\theta_2) - c_2(\theta_2)).
\end{aligned}
$$

We now turn to define the solution concepts that will be adopted in the following analyses. To start off, we first introduce some notations and terminologies. Given a protocol $m = (w,t)$,

[5] Thus, breakdowns are costly and hence inefficient.

non-empty Borel sets $\Omega_1 \subseteq \Theta_1$, $\Omega_2 \subseteq \Theta_2$, define the conditional interim payoffs as follows:

$$U_1(\theta_1', \theta_1|m, \Omega_2) := \mathbb{E}_{G_2}[(1 - w(\theta_1', \theta_2))t(\theta_1', \theta_2) + w(\theta_1', \theta_2)(p(\theta_1, \theta_2) - c_1(\theta_1))|\theta_2 \in \Omega_2]$$
$$U_2(\theta_2', \theta_2|m, \Omega_1) := \mathbb{E}_{G_1}[(1 - w(\theta_1, \theta_2'))(1 - t(\theta_1, \theta_2')) + w(\theta_1', \theta_2')(1 - p(\theta_1, \theta_2) - c_2(\theta_2))|\theta_1 \in \Omega_1].$$

Let $U_i(\cdot, \cdot|m) := U_i(\cdot, \cdot|m, \Theta_{-i})$ for each $i \in \{1, 2\}$, $U_i^*(\theta_i|m, \Omega_{-i}) := U_i(\theta_i, \theta_i|m, \Omega_{-i})$ for any non-empty Borel set $\Omega_{-i} \subseteq \Theta_{-i}$ and any $i \in \{1, 2\}$ and let $U_i^*(\cdot|m) := U_i^*(\cdot|m, \Theta_{-i})$ for each $i \in \{1, 2\}$.[6] In words, $U_i(\theta_i', \theta_i|m, \Omega_{-i})$ is the expected payoff for i under the protocol m, when having signal θ_i, reporting θ_i', given that the other is reporting truthfully and that i knows the realization of θ_{-i} is in Ω_{-i}. Notice that when $\Omega_{-i} = \Theta_{-i}$, all the notions become those under the standard interim payoffs. The following definition defines various forms of incentive compatibility formally.

Definition 2. For each $i \in \{1, 2\}$, for any non-empty Borel set $\Omega_{-i} \subseteq \Theta_{-i}$, a protocol m is Ω_{-i} *incentive compatible* for i if $U_i^*(\theta_i|m, \Omega_{-i}) \geq U_i(\theta_i', \theta_i|m, \Omega_{-i})$, $\forall \theta_i, \theta_i' \in \Theta_i$. Moreover, a protocol is *incentive compatible* for i if it is Θ_{-i}-incentive compatible for i, and is said to be *ex-post incentive compatible* for i if it is $\{\theta_{-i}\}$-incentive compatible for i for each $\theta_{-i} \in \Theta_{-i}$. Finally, it is *incentive compatible* if it is incentive compatible for each $i \in \{1, 2\}$.

On the other hand, besides incentive compatibility, the *voluntary agreement* assumption and the revelation principle requires that the equilibrium payoff under such truth-telling equilibria in direct protocols must be no less than one's payoff when bargaining breakdowns, since one can always choose to terminate the bargaining unilaterally. This then gives the *individual rationality* constraints in this setting. For notational convenience let $p_1(\theta_1, \theta_2) := p(\theta_1, \theta_2)$ and $p_2(\theta_2, \theta_1) := 1 - p(\theta_1, \theta_2)$ for all $\theta_1 \in \Theta_1, \theta_2 \in \Theta_2$. The following definition then describes various individual rationality constraints.

Definition 3. For each $i \in \{1, 2\}$, for any non-empty Borel subset $\Omega_{-i} \subseteq \Theta_{-i}$, a protocol m is Ω_{-i}-*individually rational* for i if $U_i^*(\theta_i|m, \Omega_{-i}) \geq \mathbb{E}_{G_{-i}}[p_i(\theta_i, \theta_{-i}) - c_i(\theta_i)|\theta_{-i} \in \Omega_{-i}]$, $\forall \theta_i \in \Theta_i$. Moreover, m is *individually rational* for i if it is Θ_{-i}-individually rational for i, and is said to be *ex-post individually rational* for i if it is $\{\theta_{-i}\}$-individually rational for i for any $\theta_{-i} \in \Theta_{-i}$. Finally, it is *individually rational* if it is individually rational for each $i \in \{1, 2\}$.

For terminological conveniences, we say that a protocol m is *incentive feasible* if it is both incentive compatible and individually rational. Also, for any non-empty Borel set $\Omega_i \subseteq \Theta_i$, we say

[6]As conventional, for any $i \in \{1, 2\}$, "$-i$" denotes the element $j \in \{1, 2\}$ such that $j \neq i$.

that a protocol m is Ω_i-*incentive feasible* for i if it is incentive feasible, Ω_i-incentive compatible and Ω_i-individually rational for $-i$.

As remarked above, when selecting the optimal protocol, the party with full bargaining power would face several constraints—the incentive compatibility and individual rationality constraints follow from the revelation principle and the voluntary agreement assumption. Moreover, there would be a subtle trade-off for this principal. Since payoffs when bargaining breaks down depend on private information, optimal protocols for different realizations of signals might be different. However, if the principal, given a realized signal, select the best protocol for this realization, such selection will inevitably reveal some information about the principal's private signal, which might further change the incentive constraints and might also worsen the equilibrium payoff that the principal can obtain under such protocol even though it is still feasible. In order to formally incorporate this trade-off into the decision of the principal and draw some reasonable predictions, we will adopt the solution concepts proposed in Myerson (1983). The two solution concepts that will be used below are the *strong solutions* and the *core*. In words, a strong solution for the principal is a protocol that is undominated by any other incentive feasible protocols and is ex-post incentive compatible and individually rational for the other player. On the other hand, a protocol is in the core if it is incentive feasible and cannot be *blocked* by any subset of types of the principal. The formal definitions are as follow:

Definition 4. A protocol m is a strong solution for $i \in \{1, 2\}$ if it is:

1. undominated, i.e. there is no incentive feasible protocol m' such that $U_i^*(\theta_i|m) \leq U_i^*(\theta_i|m')$, with strict inequality holding for some $\theta_i \in \Theta_i$ and

2. safe, i.e. m is incentive feasible, ex-post incentive compatible and ex-post individually rational for $-i$.

Definition 5. A protocol m is in the core for $i \in \{1, 2\}$ if it is:

1. incentive feasible and

2. cannot be blocked, i.e. there is no other protocol m' such that

$$\{\theta_i \in \Theta_i | U_i^*(\theta_i|m') > U_i^*(\theta_i|m)\} \neq \emptyset$$

and is Ω_i-incentive feasible for $-i$ for any Borel set Ω_i such that

$$\{\theta_i \in \Theta_i | U_i^*(\theta_i|m') > U_i^*(\theta_i|m)\} \subseteq \Omega_i \subseteq \Theta_i.$$

Whenever player i is assigned with full bargaining power, the strong solution, if it exists, is the most reasonable "optimal" choices of protocol that incorporate the trade-off between optimality and information revelation. The following proposition formally states this result.

Proposition 1 (Myerson, 1983)**.** *Suppose that m is a strong solution for i. Let $\hat{m}$ be any other protocol and $\Omega := \{\theta_i \in \Theta_i | U_i^*(\theta_i|m) < U_i^*(\theta_i|\hat{m})\}$. If $\Omega \neq \emptyset$, then m' is not Ω-incentive feasible for i. Furthermore, for any other strong solution m', $U_i^*(\theta_i|m) = U_i^*(\theta_i|m')$, for all $\theta_i \in \Theta_i$.*

Proof. For any i, let $m = (w, t)$ be a strong solution for i and let $\hat{m} = (\hat{w}, \hat{t})$ be any protocol such that $\Omega := \{\theta_i \in \Theta_i | U_i^*(\theta_i|m) < U_i^*(\theta_i|\hat{m})\} \neq \emptyset$. Suppose that $\hat{m}$ is Ω-incentive feasible for i. Define $\tilde{m} = (\tilde{w}, \tilde{t})$ by:

$$\tilde{w}(\theta) := \begin{cases} w(\theta), & \text{if } \theta_i \in \Theta_i \backslash \Omega \\ \hat{w}(\theta), & \text{if } \theta_i \in \Omega \end{cases} . \qquad \tilde{t}(\theta) := \begin{cases} t(\theta), & \text{if } \theta_i \in \Theta_i \backslash \Omega \\ \hat{t}(\theta), & \text{if } \theta_i \in \Omega \end{cases} .$$

Then, since m is safe and $\hat{m}$ is Ω-incentive feasible for i, the combination $\tilde{m}$ is incentive feasible. Moreover, since $\Omega \neq \emptyset$, $\tilde{m}$ dominates m, a contradiction.

For the second assertion, let m' be any other strong solution. Since m' and m are safe, they are both Ω-incentive feasible for i for any non-empty Borel subset Ω. The first assertion then implies that $\{\theta_i \in \Theta_i | U_i^*(\theta_i|m) > U_i^*(\theta_i|m')\} = \{\theta_i \in \Theta_i | U_i^*(\theta_i|m') > U_i^*(\theta_i|m)\} = \emptyset$ and thus $U_i^*(\cdot|m) \equiv U_i^*(\cdot|m')$. This completes the proof. ■

The above proposition is essentially saying that, whenever i is the principal and m is a strong solution for i, given any realization of θ_i, any other protocol that can enable i to obtain higher expected payoff must be infeasible after the revelation of information by such selection. Therefore, a strong solution, provided that it exists, will be the most reasonable protocol for the principal to choose, since it perfectly balances the trade-off described above. Furthermore, as it will be seen to be useful in later results, any two strong solutions must be interim payoff equivalent for the principal so there would be no loss of generality to restrict attentions to just one of the strong solutions.

However, a major drawback of strong solutions is that they might not always exist. Myerson (1983) accommodates the non-existing problem by introducing an axiomatically-defined solution concept, called the *neutral mechanisms*, and shows that there always exists neutral mechanism and any strong solution must be a neutral mechanism when it exists. Nevertheless, solving for the neutral mechanisms explicitly in our setting may be challenging. Instead, we exploit another property of the neutral mechanism—that it must be a subset of the *core*—and derive properties that hold for every mechanism in the core. As defined, an element in the core is a protocol that is immune to any potential fictitious "blocking coalition" formed by some types of the principal. If a protocol is not in the core, it means that there will be some realizations for the principal's types under which the principal can choose another protocol that would yield higher payoff and is also feasible after the information revelation induced by such choice.[7]

Finally, we further introduce some terminologies and notations that will be used throughout the following analyses. Given any two protocols $m = (w, t)$ and $m' = (w', t')$, we say that m is *essentially equivalent* to m' if $w \equiv w'$ (Lebesgue) almost everywhere (on their domains relevant to the context) and $t \equiv t'$ (Lebesgue) almost everywhere whenever $w \neq 1$. We also say that the protocols satisfying some property is *essentially unique* if whenever m, m' satisfy such property, m and m' are essentially equivalent. For notational conveniences, we will write $\mathbf{0}$ and $\mathbf{1}$ as constant functions taking value 0 and 1, respectively. With these definitions and solution concepts, we are now able to analyze the crisis bargaining problems with one party having full bargaining power under various information structures.

3 Crisis Bargaining with Private Values

In this section, we first study the case of private value. Namely, cases in which one's private information is about elements that only affects her own payoff. In other words, the cases where uncertainties are about c_1 and/or c_2.

3.1 Private Information on the Principal

First, we consider the case where only the principal has private information. Formally, suppose that player 1 has full bargaining power and that c_2 and p are constant functions. Since c_1 is a function of θ_1, we can write F_1 as the CDF induced by c_1 and G_1, namely, $F_1(c_1) := \mathbb{P}_{G_1}(c_1(\theta_1) \leq c_1)$,

[7] Notice that the solution concepts in Myerson (1983), and thus the ones adopted here are essentially concepts in *cooperative game theory*. As remarked in Myerson (1983), the introduction of cooperative game theoretic solution concepts is not surprising in this setting as it captures the idea how to compromise the interests between different types of the principal under certain feasibility constraints. Unlike Maskin & Tirole (1990, 1993) and thus Leventoglu (2012), who adopted a non-cooperative approach in mechanism selections, we found that the cooperative concepts require less further specifications on the bargaining structure—and perhaps more importantly— more tractable than the non-cooperative approach when the types are continuous.

for all $c_1 \in [\underline{c_1}, \overline{c_1}]$ and equivalently suppose that $c_1 \sim F_1$ and $p \in [0,1]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$ are common knowledge. We further assume that F_1 admits density f_1 (with respect to the Lebesgue measure) that has full support on $[\underline{c_1}, \overline{c_1}]$. Also, to rule out trivial solutions, assume also that $1 - p - c_2 \geq 0$. These assumptions are summarized below.

Assumption 1. *F_1 admits density f_1 with $f_1 > 0$ on $[\underline{c_1}, \overline{c_1}]$ and $1 - p - c_2 \geq 0$.*

Therefore, the general setting in the previous section can be translated into that with a protocol being $m = (w, t) : [\underline{c_1}, \overline{c_1}] \to [0,1]^2$. Under any such protocol, when player 1 is has realized type c_1 and reports c_1', (ex-post) payoffs will respectively be:

$$(1 - w(c_1'))t(c_1') + w(c_1')(p - c_1) \tag{1}$$

$$(1 - w(c_1'))(1 - t(c_1')) + w(c_1')(1 - p - c_2). \tag{2}$$

Applying the previous definitions to this environment, a protocol m is incentive feasible if

$$U_1^*(c_1|m) \geq U_1(c_1', c_1|m), \forall c_1, c_1' \in [\underline{c_1}, \overline{c_1}],$$

$$U_1^*(c_1|m) \geq p - c_1, \forall c_1 \in [\underline{c_1}, \overline{c_1}],$$

$$U_{2,m}^* \geq 1 - p - c_2,$$

where $U_{2,m}^* := U_2^*(c_2|m)$.[8] Notice also that, since the surplus is divided without loss if bargaining does not breakdown, by setting $c_1' = c_1$ and taking expectation with respect to c_1 under F_1 on (1) and (2) and then sum them up, we have:

$$\mathbb{E}_{F_1}[U_1^*(c_1|m)] + U_{2,m}^* = 1 - \mathbb{E}_{F_1}[w(c_1)(c_1 + c_2)].^9 \tag{3}$$

That is, the sum of interim payoffs depends only on w but not on t. This expression will be useful in later derivations, which will be referred as the *resource constraint* henceforth. This observation, together with the standard revenue-equivalence formula and a note that sharing rule t is bounded and can only be in effect when the bargaining does not breakdown, the following lemma gives a useful characterization for the set of incentive feasible protocols.

[8] Since c_2 is common knowledge, $U_2^*(c_2|m)$ does not depend on c_2 for any protocol m.

[9] In fact, this equation reflects the nature of crisis bargaining. As we can see in a crisis bargaining, there are both *conflict of interests* and *common interests* between two parties. The conflicts inherit in the feature of division of a fixed surplus whereas the common interest comes from the fact that war is costly and thus is beneficial for both to avoid it. The constant 1 on the right hand side and the additive relation between expected payoffs on the left hand side capture the conflict of interest while the second term on the right hand side captures the common interest.

Lemma 1. *Suppose that only c_1 is private information and $c_1 \sim F_1$ with density f_1. Given w, there exists t such that (w,t) is incentive feasible if and only if there exists $U_1^*(\underline{c_1})$, U_2^* such that:*

1. *w is non-increasing.*

2. *$U_1^*(\underline{c_1}) - \int_{\underline{c_1}}^{c_1} w(x)dx \geq p - c_1, \forall c_1 \in [\underline{c_1}, \overline{c_1}]$.*

3. *$U_2^* \geq 1 - p - c_2$.*

4. *$U_1^*(\underline{c_1}) + U_2^* = 1 - \mathbb{E}_{F_1}[w(c_1)(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)})]$.*

5. *$w(\underline{c_1})(p - \underline{c_1}) \leq U_1^*(\underline{c_1}) \leq (1 - w(\underline{c_1})) + w(\underline{c_1})(p - \underline{c_1})$.*

The proof of this lemma can be found in the appendix. Condition 1 and and 2 follow from the standard revenue-equivalence formula. The inequalities in conditions 2 and 3 follow from the individual rationality constraints. Condition 4 follows from the recourse constraint (3) and Fubini's theorem and condition 5 is from the boundedness of t. Let $\mathcal{M}_1$ denote the set of vectors $(U_1^*(\underline{c_1}), U_2^*, w)$ in the normed linear space $\mathbb{R}^2 \times L^2([\underline{c_1}, \overline{c_1}])$. Notice that condition 4 in Lemma 1 holds in equality and hence we can identify an element in $\mathcal{M}_1$ by just a constant $U_1^*(\underline{c_1})$ and a non-increasing function w satisfying several inequalities. Notice also that the set $\mathcal{M}_1$ is closed (under the strong topology) and convex,[10] and thus is (weakly) compact by Mazur's theorem and the Kakutani theorem.[11] By the equivalence, we sometimes will slightly abuse notations and use m to denote an element of $\mathcal{M}_1$. Also, we will say that t (interim) *implements* w if there exists $U_1^*(\underline{c_1}), U_2^*$ such that $(U_1^*(\underline{c_1}), U_2^*, w) \in \mathcal{M}_1$ and

$$(1 - w(c_1))t(c_1) + w(c_1)(p - c_1) = U_1^*(\underline{c_1}) - \int_{\underline{c_1}}^{c_1} w(x)dx, \forall c_1 \in [\underline{c_1}, \overline{c_1}]$$
$$\mathbb{E}_{F_1}[(1 - w(c_1))(1 - t(c_1)) + w(c_1)(1 - p - c_2)] = U_2^*.$$

Notice that by definition, if t implements w, then (w,t) is incentive feasible. With the characterization of Lemma 1, we have the main result under this information structure.

[10] To see this, first observe that condition 1 in Lemma 1 can in fact be replaced by "non-increasing" Lebesgue almost everywhere, as the proof shows. Now given any sequence $\{(U_1^{(n)}, U_2^{(n)}, w^{(n)})\}$ that converges in the L^2 norm to some (U_1, U_2, w). By the Bolzano-Wierestrass theorem and the fact that $||w^{(n)} - w||_{L^2([\underline{c_1}, \overline{c_1}])} \to 0$ implies that there exists a subsequence $\{w^{(n_k)}\}$ that converges pointwisely Lebesgue almost everywhere to w, we can take a sequence such that $\{U_1^{(n)}\} \to U_1$, $\{U_2^{(n)}\} \to U_2$ and $\{w^{(n)}\} \to w$ pointwisely almost everywhere. It is then clear that conditions 1 to 5 holds. Therefore, $\mathcal{M}_1$ is indeed strongly closed.

[11] In this case, the set $\mathcal{M}_1$ is in fact strongly compact by the Helly selection theorem and the Lebesgue dominant convergent theorem, but for our purpose here, weak compactness is enough.

Proposition 2. *Suppose that player 1 has full bargaining power, that only c_1 is private information with $c_1 \sim F_1$ and that Assumption 1 holds. Then strong solution exists. Moreover, in any strong solution $\hat{m} = (\hat{w}, \hat{t})$ for 1, $\hat{w} \equiv 0$ Lebesgue-almost everywhere. In particular, the ex-ante probability of breakdown, $\mathbb{E}_{F_1}[\hat{w}(c_1)] = 0$*

Proof. Consider first the problem

$$\max_{m \in \mathcal{M}_1} \mathbb{E}_{F_1}[U_1^*(c_1|m)]. \tag{4}$$

By Lemma 1, $\mathcal{M}_1$ is (weakly) compact and $\mathbb{E}_{F_1}[U_1^*(c_1|m)]$ is linear in m and thus problem in (4) is well-defined. Also, since f_1 has full support on $[\underline{c_1}, \overline{c_1}]$, the solution to (4) gives an undominated protocol. Using the recourse constraint (3), problem (4) can be rewritten into:

$$\max_{w(\cdot), U_2^*} \left[1 - U_2^* - \mathbb{E}_{F_1}[(w(c_1)(c_1 + c_2))]\right]$$
$$\text{s.t. } U_2^* \geq 1 - p - c_2.$$

Pointwise maximization then gives $w^* \equiv 0$, $U_{2,m^*}^* = 1 - p - c_2$ as the solution. Let $m^* := (p + c_2, 1 - p - c_2, w^*)$, from $1 - p - c_2 \geq 0$, condition 5 in Lemma 1 holds. Also, since $\underline{c_1} > 0$, $c_2 > 0$, condition 2 also holds. By construction, all the rest of the conditions also hold and thus m^* is indeed incentive feasible. Let $t^*(c_1) := p + c_2$, $\forall c_1 \in [\underline{c_1}, \overline{c_1}]$. Then t^* (interim) implements w^* and hence (w^*, t^*) is incentive feasible. Finally,

$$(1 - w^*(c_1))(1 - t^*(c_1)) + w^*(c_1)(1 - p - c_2) = 1 - p - c_2, \forall c_1 \in [\underline{c_1}, \overline{c_1}].$$

Hence, (w^*, t^*) is ex-post incentive compatible and individually rational for 2 and thus is safe.

Moreover, let $\hat{m} = (\hat{w}, \hat{t})$ be any strong solution. Proposition 1 gives that $U_1^*(c_1|m^*) = U_1^*(c_1|\hat{m})$, for all $c_1 \in [\underline{c_1}, \overline{c_1}]$. Also, since $\hat{m}$ is incentive feasible, Lemma 1 then implies that

$$U_1^*(c_1|\hat{m}) = U_1^*(\underline{c_1}|\hat{m}) - \int_{\underline{c_1}}^{c_1} \hat{w}(x)dx, \forall c_1 \in [\underline{c_1}, \overline{c_1}].$$

Together, for (Lebesgue) almost all $c_1 \in [\underline{c_1}, \overline{c_1}]$,[12]

$$\hat{w}(c_1) = U_1^{*'}(c_1|\hat{m}) = U_1^{*'}(c_1|m^*) = w^*(c_1) = 0.$$

Finally, since F_1 is absolute continuous, $\hat{w} \equiv 0$ F_1-almost everywhere as well. Therefore, $\mathbb{E}_{F_1}[\hat{w}(c_1)] = 0$ This completes the proof. ■

[12]That is, for all $c_1 \in [\underline{c_1}, \overline{c_1}]$ such that $U_1^*(\cdot|\hat{m})$ and $U_1^*(\cdot|m^*)$ are differentiable.

To better understand this result, let us fisrt recall the essential trade-off problem for the principal when selecting protocols. On one hand, the principal wishes to choose the protocol that benefits her the most for the given realization c_1. On the other hand, such selection cannot reveal information to a degree that it actually does more harm than benefits. Strong solution, by its definition and by Proposition 1, is the most reasonable criterion to reconcile this conflict. By invoking the supporting hyperplane theorem and Lemma 1, we can see that a protocol $m \in \mathcal{M}_1$ is undominated if and only if there exists a "weight" $\lambda : [\underline{c_1}, \overline{c_1}] \to \mathbb{R}$ such that

$$\int_{\underline{c_1}}^{\overline{c_1}} \lambda(c_1) U_1^*(c_1|m) dc_1 \geq \int_{\underline{c_1}}^{\overline{c_1}} \lambda(c_1) U_1^*(c_1|m') dc_1, \forall m' \in \mathcal{M}_1.$$

As such, a strong solution can then be thought of as a protocol that maximizes the principal's weighed sum of payoff under certain weights that reconcile different objectives of different types in a way that is consistent with the subordinate's available information and incentives. It turns out that, as shown in the proof of Proposition 2, if the "weights" exactly reflect the subordinate's ex-ante belief, namely, the "weight" is given by density f_1, then the desired consistency can be achieved. Intuitively, this is rather clear since the protocols are selected among the incentive feasible ones, which is in the interim sense for the subordinate under F_1, by taking weights that reflect exactly such belief, the optimal solution would clearly be consistent with the available information and incentives. Another noteworthy feature is that, the protocol m^* being safe is relying on the private value environment. Since the subordinate's payoff when bargaining breakdowns does not depend on the principal's private information, ex-post individual rationality constraint is rather easier to be satisfied. This feature will play the same role for all the other information structures under private value environment but will no longer be true as we move to common value environments.

3.2 Private Information on the Subordinate

We now turn to study the case in which only the subordinate has private information. Similarly, since c_2 is a function of θ_2, it is without loss of generality to suppose that $c_2 \sim F_2$ for some CDF F_2 and that only player 2 can observe its realization. Again, suppose that player 1 has full bargaining power and that $p \in [0, 1]$, $c_1 \in [\underline{c_1}, \overline{c_1}]$ are common knowledge. In this setting, notice that the player with full bargaining power does not have full information and thus allocation of bargaining power and information structure are *misaligned*. As in the previous case, we will impose some assumptions of F_2, p and $\underline{c_2}$. This is summarized below:

Assumption 2. *F_2 admits density f_2 and has full support on $[\underline{c_2}, \overline{c_2}]$. Moreover, F_2 has increasing* *hazard rate. Namely, the function* $c_2 \mapsto \frac{f_2(c_2)}{1-F_2(c_2)}$ *is increasing. Also, assume that* $1 - p - \underline{c_2} > 0$.

Ex-post payoffs under any protocol $m = (w,t)$ in this environment is similar to the previous case, except that it is now player 2 making reports and thus the domain of w and t becomes $[\underline{c_2}, \overline{c_2}]$.[13] From the definitions in the previous section, a protocol m is incentive feasible if

$$\begin{aligned}
&U^*_{1,m} \geq p - c_1, \\
&U^*_2(c_2|m) \geq U_1(c'_2, c_2|m), \forall c_2, c'_2 \in [\underline{c_2}, \overline{c_2}], \\
&U^*_2(c_2|m) \geq 1 - p - c_2, \forall c_2 \in [\underline{c_1}, \overline{c_1}],
\end{aligned}$$

where $U^*_{1,m} = U^*_1(c_1|m)$, $\forall c_1$. Similarly, the resource constraint is given by

$$\mathbb{E}_{F_2}[U^*_2(c_2|m)] + U^*_{1,m} = 1 - \mathbb{E}_{F_2}[w(c_2)(c_1 + c_2)]. \tag{5}$$

As in Lemma 1 the following lemma characterizes the set of incentive feasible protocols.

Lemma 2. *Suppose that only* c_2 *is private information and* $c_2 \sim F_2$ *with density* f_2*. Given* w*, there exists* t *such that* (w,t) *is incentive feasible if and only if there exists* $U^*_2(\underline{c_2})$, U^*_1 *such that:*

1. w *is non-increasing.*
2. $U^*_2(\underline{c_2}) - \int_{\underline{c_2}}^{c_2} w(x)dx \geq 1 - p - c_2, \forall c_2 \in [\underline{c_2}, \overline{c_2}]$.
3. $U^*_1 \geq p - c_1$.
4. $U^*_2(\underline{c_2}) + U^*_1 = 1 - \mathbb{E}_{F_2}[w(c_2)(c_1 + c_2 - \frac{1-F_2(c_2)}{f_2(c_2)})]$.
5. $w(\underline{c_2})(1 - p - \underline{c_2}) \leq U^*_2(\underline{c_2}) \leq (1 - w(\underline{c_2})) + w(\underline{c_2})(1 - p - \underline{c_2})$.

Lemma 2 is essentially the same as Lemma 1, except that the role of subscripts 1 and 2 are interchanged. In the same way, let $\mathcal{M}_2$ denote the set of vectors $(U^*_1, U^*_2(\underline{c_2})), w(\cdot))$ in the normed linear space $\mathbb{R}^2 \times L^2([\underline{c_2}, \overline{c_2}])$. that satisfies the conditions in Lemma 2. As above, it is clear that the set $\mathcal{M}_2$ is convex and (strongly) closed and thus is (weakly) compact. Given w, we will say that a function $t : [\underline{c_2}, \overline{c_2}] \to [0,1]$ implements w if there exists $U^*_1, U^*_2(\underline{c_2})$ such that $(U^*_1, U^*_2(\underline{c_2}), w) \in \mathcal{M}_2$ and

$$\begin{aligned}
&\mathbb{E}_{F_2}[(1 - w(c_2))t(c_2) + w(c_2)(p - c_1)] = U^*_1 \\
&(1 - w(c_2))(1 - t(c_2)) + w(c_2)(1 - p - c_2) = U^*_2(\underline{c_2}) - \int_{\underline{c_2}}^{c_2} w(x)dx, \forall c_2 \in [\underline{c_2}, \overline{c_2}],
\end{aligned}$$

[13] Therefore, by substituting c'_1 with c'_2 in (1) and (2), we have the ex-post payoffs.

and we will also say that w is implemented by t with U_2 if such $U_2^*(\underline{c_2}) = U_2$.[14] From Lemma 2, we can see that if t implements w, then (w,t) is incentive feasible. Conversely, for any $(U_1^*, U_2^*(\underline{c_2}), w) \in \mathcal{M}_2$, there exists a $t : [\underline{c_2}, \overline{c_2}] \to [0,1]$ that implements w with $U_2^*(\underline{c_2})$. Since player 1 does not have private information, solving for strong solutions reduces to solving a constraint optimization problem. Together with Lemma 2, this gives the main result under such information structure.

Proposition 3. *Suppose that only c_2 is private information with $c_2 \sim F_2$, that player 1 has full bargaining power, and that Assumption 2 holds. Then strong solution exists and in any strong solution (w^*, t^*), $w^*(c_2) = 1$ if $c_2 < c_2^*$ and $w^*(c_2) = 0$ if $c_2 \geq \underline{c_2}$, where c_2^* is (well) defined as follows:*

$$c_2^* = \frac{1 - F_2(c_2^*)}{f_2(c_2^*)} - c_1, \tag{6}$$

if $\underline{c_2} - \frac{1}{f_2(\underline{c_2})} \leq -c_1$ and $c_2^* = \underline{c_2}$ otherwise. Moreover, the strong solution is essentially unique.

Proof. We first verify that c_2^* is well-defined. Indeed, by the increasing hazard rate assumption, for given c_1, the function $c_2 \mapsto c_2 - \frac{1-F_2(c_2)}{f_2(c_2)}$ is increasing and thus there exists a unique $c_2^* \in [\underline{c_2}, \overline{c_2}]$ that solves (6) whenever $\underline{c_2} - \frac{1}{f_2(\underline{c_2})} \leq -c_1$. Thus, such c_2^* is indeed well-defined. Now Consider the problem

$$\max_{m \in \mathcal{M}_2} U_{1,m}^*. \tag{7}$$

Since $\mathcal{M}_2$ is (weakly) compact and $U_{1,m}^*$ is linear in m by Lemma 2, problem (7) is well-defined. As remarked, since 1 does not have private information, m^* is a undominated and safe for 1 if and only if m^* solves problem (7). By using conditions 1 to 4 of Lemma 2, problem (7) can be solved by considering:

$$\begin{aligned} \max_{U_2^*(\underline{c_2}), w(\cdot)} & \left[1 - U_2^*(\underline{c_2}) - \mathbb{E}_{F_2}\left[w(c_2)\left(c_1 + c_2 - \frac{1 - F_2(c_2)}{f_2(c_2)}\right)\right]\right] \\ \text{s.t. } & w \text{ non-increasing}, \\ & U_2^*(\underline{c_2}) \geq 1 - p - \underline{c_2}, \end{aligned} \tag{8}$$

If the solution to problem (8) also satisfies condition 5 in Lemma 2, then it is a solution of problem (7). By the increasing hazard rate assumption, for given c_1, the function $c_2 \mapsto c_1 + c_2 - \frac{1-F_2(c_2)}{f_2(c_2)}$ is increasing. Pointwise maximization then shows that w^*, together with $U_2^*(\underline{c_2}|m^*) := 1-p-\underline{c_2}$, solves problem (8). Together with the assumption that $1-p-\underline{c_2} > 0$, let $U_{1,m^*}^* := p+\underline{c_2} - \mathbb{E}_{F_2}[w^*(c_2)(c_1+$

[14]Since from the recourse constraint, and hence condition 4 in Lemma 2, it suffices to identity elements in $\mathcal{M}_2$ by a constant U_2 and a function w. Therefore, the terminology here considers only one constant.

$c_2 - \frac{1-F_2(c_2)}{f_2(c_2)})]$, conditions 1-5 in Lemma 2 are satisfied under $m^* := (U^*_{1,m^*}, U^*_2(\underline{c_2}|m^*), w^*)$ and hence $m^* \in \mathcal{M}_2$. This shows that m^* solves problem (7) and hence is a strong solution. Furthermore, let $\hat{m} = (\hat{w}, \hat{t})$ be any strong solution. Then $(1 - p - \underline{c_2}, \hat{w})$ must be a solution to (8), since $\hat{w}$ must be non-increasing, $\hat{w}(c_2) = w^*(c_2)$, $\forall c_2 \neq c_2^*$. From Proposition 1, $U^*_{1,\hat{m}} = U^*_{1,m^*}$. Together with condition 4 in Lemma 2, $U^*_2(\underline{c_2}|m^*) = U_2(\underline{c_2}|\hat{m})$. By condition 2, the envelope characterization, we must then have $U^*_2(c_2|m^*) = U^*_2(c_2|\hat{m})$ for (Lebesgue) almost all $c_2 \in [\underline{c_2}, \overline{c_2}]$. Together, $\hat{t} \equiv t^*$ Lebesgue almost everywhere whenever $\hat{w} < 1$. ■

Several features are noteworthy in this proposition. First notice that whenever the density f_2 has small likelihoods on the left tail or $\underline{c_2}$ is small enough, the condition $\underline{c_2} - \frac{1}{f_2(\underline{c_2})} < -c_1$ holds and thus $c_2^* > \underline{c_2}$. This then implies that given c_1, in the (essentially unique) strong solution, probability of breakdown, $\mathbb{E}_{F_2}[w^*(c_2)]$, unlike the case when 1 has full information, is positive and therefore there will be efficiency loss. Second, as in the previous case. The outcome in the strong solution is exactly the same as the equilibrium in the motivating example. Finally, for the purposes in the next subsection, although c_1 is common knowledge in this case, we can still think of c_1 as being drawn from the CDF F_1 but is perfectly observed by both parties. As such, c_2^* given in this proposition is essentially a function of realizations c_1. We can write it as $c_2^*(c_1)$, that is, for each c_1

$$c_2^*(c_1) = \frac{1 - F_2(c_2^*(c_1))}{f_2(c_2^*(c_1))} - c_1,$$

if $\underline{c_2} - \frac{1}{f_2(\underline{c_2})} \leq -c_1$ and $c_2^*(c_1) = \underline{c_2}$, otherwise. Also write, for each realization of c_1, the corresponding optimal solutions to problem (7) as $m^*(c_1) = (U^*_{1,m^*(c_1),c_1}, U^*_2(\underline{c_2}|m^*(c_1), c_1), w^*(\cdot|c_1))$ By the proposition above $m^*(c_1) \in \mathcal{M}_2(c_1)$ and hence exists $t^*(\cdot|c_1)$ that implements $w^*(\cdot|c_1)$ with $U^*_2(\underline{c_2}|m^*(c_1), c_1)$ for each $c_1 \in [\underline{c_1}, \overline{c_1}]$. Therefore, for each c_1, $(w^*(\cdot|c_1), t^*(\cdot|c_1))$ is the (essentially unique) strong solution for 1 under c_1. The ex-ante probability of breakdown in the strong solution, by the increasing hazard rate assumption and Proposition 3 is then

$$\mathbb{E}[w^*(c_2|c_1)] = \mathbb{P}\left(c_1 + c_2 - \frac{1 - F_2(c_2)}{f_2(c_2)} < 0\right),$$

which will be positive provided that $\underline{c_1} + \underline{c_2} < \frac{1}{f_2(\underline{c_2})}$, where the expectation and probability are both taken under the joint distribution F_1F_2.

3.3 Private Information on Both Parties

Finally, we examine the information structure under which both parties have private information. Again, from the facts that c_1 and c_2 are functions of θ_1 and θ_2 respectively, we may without loss of generality suppose that $c_1 \sim F_1$, $c_2 \sim F_2$ for CDFs F_1 and F_2. Suppose that player 1 has full bargaining power and that $p \in [0,1]$ is common knowledge. Similarly, we impose some assumptions on F_1, F_2 and $p, \underline{c_2}$.

Assumption 3. *F_1, F_2 admit densities f_1, f_2, respectively and have full support and increasing hazard rate. Moreover, $1 - p - \underline{c_2} > 0$, $p - \underline{c_1} > 0$.*[15]

In this environment, a protocol is a pair of functions $(w, t) : [\underline{c_1}, \overline{c_1}] \times [\underline{c_2}, \overline{c_2}] \to [0,1]^2$. Under any protocol (w, t), when 1 and 2 have types c_1, c_2 and report c_1', c_2' respectively, payoffs are respectively:

$$(1 - w(c_1', c_2'))t(c_1', c_2') + w(c_1', c_2')(p - c_1) \tag{9}$$

$$(1 - w(c_1', c_2'))(1 - t(c_1', c_2')) + w(c_1', c_2')(1 - p - c_2). \tag{10}$$

Also, from the definition in the previous section, a protocol m is incentive feasible if

$$\begin{aligned}
&U_1^*(c_1|m) \geq U_1(c_1', c_1|m), \forall c_1, c_1' \in [\underline{c_1}, \overline{c_1}],\\
&U_1^*(c_1|m) \geq p - c_1, \forall c_1 \in [\underline{c_1}, \overline{c_1}],\\
&U_2^*(c_2|m) \geq U_2(c_2', c_2|m), \forall c_2', c_2 \in [\underline{c_2}, \overline{c_2}],\\
&U_2^*(c_2|m) \geq 1 - p - c_2, \forall c_2 \in [\underline{c_2}, \overline{c_2}].
\end{aligned}$$

For any protocol $m = (w, t)$, the recourse constraint in this environment is then given by:

$$\mathbb{E}_{F_1}[U_1^*(c_1|m)] + \mathbb{E}_{F_2}[U_2^*(c_2|m)] = 1 - \mathbb{E}[w(c_1, c_2)(c_1 + c_2)]. \tag{11}$$

In addition, as conventional, denote the interim reduced forms as:

$$\begin{aligned}
\bar{w}_1(c_1) &:= \int_{\underline{c_2}}^{\overline{c_2}} w(c_1, c_2) dF_2(c_2), \forall c_1 \in [\underline{c_1}, \overline{c_1}],\\
\bar{w}_2(c_2) &:= \int_{\underline{c_1}}^{\overline{c_1}} w(c_1, c_2) dF_1(c_1), \forall c_2 \in [\underline{c_2}, \overline{c_2}].
\end{aligned}$$

As the above two cases, by the standard envelope argument, the recourse constraint (11), and noticing that t is bounded and only enters in effect when $w \neq 1$. The set of incentive feasible protocols can be characterized as in the following lemma.

[15] In fact, the increasing hazard rate assumption for F_1 and the assumption that $p - \underline{c_1} > 0$ is not necessary for the result here. However, they allow us to describe the strong solution for both parties more compactly, which will be useful in the later section when intermediate allocation of bargaining power is considered.

Lemma 3. *Suppose that only c_1 and c_2 are private information and that $c_1 \sim F_1$, $c_2 \sim F_2$, with densities f_1 and f_2, respectively. Given w, there exists t such that (w,t) is incentive feasible if and only if there exists $U_1^*(\underline{c_1})$, $U_2^*(\underline{c_2})$ such that:*

1. *$\bar{w}_1$ and $\bar{w}_2$ are non-increasing.*
2. *$U_1^*(\underline{c_1}) - \int_{\underline{c_1}}^{c_1} \bar{w}_1(x)dx \geq p - c_1, \forall c_1 \in [\underline{c_1}, \overline{c_1}]$.*
3. *$U_2^*(\underline{c_2}) - \int_{\underline{c_2}}^{c_2} \bar{w}_2(x)dx \geq 1 - p - c_2, \forall c_2 \in [\underline{c_2}, \overline{c_2}]$.*
4. *$U_1^*(\underline{c_1}) + U_2^*(\underline{c_2}) = 1 - \mathbb{E}[w(c_1,c_2)(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} - \frac{1-F_2(c_2)}{f_2(c_2)})]$.*
5. *$\bar{w}_1(\underline{c_1})(p - \underline{c_1}) \leq U_1^*(\underline{c_1}) \leq (1 - \bar{w}_1(\underline{c_1})) + \bar{w}_1(\underline{c_1})(p - \underline{c_1})$.*
6. *$\bar{w}_2(\underline{c_2})(1 - p - \underline{c_2}) \leq U_2^*(\underline{c_2}) \leq (1 - \bar{w}_2(\underline{c_2})) + \bar{w}_2(\underline{c_2})(1 - p - \underline{c_2})$.*

Let $\mathcal{M}_3$ denote the set of vectors $(U_1^*(\underline{c_1}), U_2^*(\underline{c_2}), w(\cdot,\cdot))$ in the normed linear space $\mathbb{R}^2 \times L^2([\underline{c_1}, \overline{c_1}] \times [\underline{c_2}, \overline{c_2}])$ that satisfies the conditions in Lemma 3. As before, $\mathcal{M}_3$ is clearly (strongly) closed and convex and hence is (weakly) compact. Similarly, we will say that a function $t : [\underline{c_1}, \overline{c_1}] \times [\underline{c_2}, \overline{c_2}] \to [0,1]$ implements w if there exists $U_1^*(\underline{c_1}), U_2^*(\underline{c_2})$ such that $(U_1^*(\underline{c_1}), U_2^*(\underline{c_2}), w) \in \mathcal{M}_3$ and

$$\mathbb{E}_{F_2}[(1 - w(c_1,c_2))t(c_1,c_2) + w(c_1,c_2)(p - c_1)] = U_1^*(\underline{c_1}) - \int_{\underline{c_1}}^{c_1} \bar{w}_1(x)dx, \forall c_1 \in [\underline{c_1}, \overline{c_1}],$$

$$\mathbb{E}_{F_1}[(1 - w(c_1,c_2))(1 - t(c_1,c_2)) + w(c_1,c_2)(1 - p - c_2)] = U_2^*(\underline{c_2}) - \int_{\underline{c_2}}^{c_2} \bar{w}_2(x)dx, \forall c_2 \in [\underline{c_2}, \overline{c_2}].$$

We also say that w is implemented by t with U_2 if such $U_2^*(\underline{c_2}) = U_2$. By Lemma 3, if t implements w, then (w,t) is incentive feasible. Conversely, for any $(U_1^*(\underline{c_1}), U_2^*(\underline{c_2}), w) \in \mathcal{M}_3$, there exists $t : [\underline{c_1}, \overline{c_1}] \times [\underline{c_2}, \overline{c_2}] \to [0,1]$ such that t implements w. Using this characterization, we can again solve for the strong solution under such information structure, which is summarized in the following proposition. Before we begin the discussion of the next proposition, we first recall some results from the previous subsection. Recall that at the end of the previous subsection, we used the notations $m^*(c_1) = (U_{1,m^*(c_1),c_1}^*, U_2^*(\underline{c_2}|m^*(c_1),c_1), w^*(\cdot|c_1))$ to denote the optimal incentive feasible protocol for 1 given a fixed $c_1 \in [\underline{c_1}, \overline{c_1}]$. By Proposition 3 and the increasing hazard rate assumption, we can see that

$$w^*(c_2|c_1) = \begin{cases} 1, & \text{if } c_1 + c_2 - \frac{1-F_2(c_2)}{f_2(c_2)} \\ 0, & \text{otherwise} \end{cases}, \forall c_1 \in [\underline{c_1}, \overline{c_1}], c_2 \in [\underline{c_2}, \overline{c_1}],$$

and $U_2^*(\underline{c_2}|m^*(c_1),c_1) = 1 - p - \underline{c_2}$, for any $c_1 \in [\underline{c_1}, \overline{c_1}]$. Notice also that $m^*(c_1) \in \mathcal{M}_2(c_1)$ for each $c_1 \in [\underline{c_1}, \overline{c_1}]$. Thus, there exists some $t^*(\cdot|c_1)$ that implements $w^*(\cdot|c_1)$ with $U_2^*(\underline{c_2}|m^*(c_1),c_1)$

for each $c_1 \in [\underline{c_1}, \overline{c_1}]$. Optimality of problem (7) also implies $U^*_{1,m^*(c_1)^*,c_1} \geq U^*_{1,m^*(c_1'),c_1}$, for any $c_1', c_1 \in [\underline{c_1}, \overline{c_1}]$. These observations will be useful for the proof of the following proposition.

Proposition 4. *Suppose that only c_1 and c_2 are private information with $c_1 \sim F_1$, $c_2 \sim F_2$, that player 1 has full bargaining power and that Assumption 3 holds. Then strong solution exists. Moreover, in any strong solution $m = (w,t)$, $\bar{w}_1 \equiv \bar{w}_1^*$ Lebesgue almost everywhere on $[\underline{c_1}, \overline{c_1}]$, where*

$$w^*(c_1, c_2) = \begin{cases} 1, & \text{if } c_1 + c_2 - \frac{1-F_2(c_2)}{f_2(c_2)} < 0 \\ 0, & \text{otherwise} \end{cases} . \tag{12}$$

In particular, in any strong solution $m = (w,t)$, $\mathbb{E}[w(c_1,c_2)] = \mathbb{E}[w^*(c_2|c_1)]$, where the expectation is taken under joint distribution F_1F_2.[16]

Proof. Consider first the problem

$$\max_{m \in \mathcal{M}_3} \mathbb{E}_{F_1}[U_1^*(c_1|m)]. \tag{13}$$

Since $\mathcal{M}_3$ is (weakly) compact and $\mathbb{E}_{F_1}[U_1^*(c_1|m)]$ is linear in m by Lemma 3, the problem in (13) is well-defined. Furthermore, by using condition 4 and the envelope characterization in condition 2 in Lemma 3, for any $m \in \mathcal{M}_3$, $c_1 \in [\underline{c_1}, \overline{c_1}]$,

$$U_1^*(c_1|m) = 1 - U_2^*(\underline{c_2}|m) - \mathbb{E}\left[w(c_1,c_2)\left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} - \frac{1-F_2(c_2)}{f_2(c_2)}\right)\right] - \int_{\underline{c_1}}^{c_1} \bar{w}_1(x)dx. \tag{14}$$

Taking expectation with respect to c_1 under F_1 on both sides, and apply Fubini's theorem again, for any $m \in \mathcal{M}_3$, we have:

$$\mathbb{E}_{F_1}[U_1^*(c_1|m)] = 1 - U_2^*(\underline{c_2}|m) - \mathbb{E}\left[w(c_1,c_2)\left(c_1 + c_2 - \frac{1-F_2(c_2)}{f_2(c_2)}\right)\right]. \tag{15}$$

Therefore, problem (13) can be solved by considering:

$$\begin{aligned} &\max_{U_2^*(\underline{c_2}), w(\cdot,\cdot)} \left[1 - U_2^*(\underline{c_2}|m) - \mathbb{E}\left[w(c_1,c_2)\left(c_1 + c_2 - \frac{1-F_2(c_2)}{f_2(c_2)}\right)\right]\right], \\ &\quad \text{s.t. } U_2^*(\underline{c_2}) \geq 1 - p - \underline{c_2}, \\ &\quad\quad \bar{w}_1, \bar{w}_2 \text{ are non-increasing,} \end{aligned} \tag{16}$$

Given a solution of (16), say, (U_2, w). If such w can be can be implemented by some function $t : [\underline{c_1}, \overline{c_1}] \times [\underline{c_2}, \overline{c_2}] \to [0,1]$ with U_2, then by Lemma 3, such protocol must be incentive feasible. Let $U_1 := 1 - U_2 - \mathbb{E}[w(c_1,c_2)(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} - \frac{1-F_2(c_2)}{f_2(c_2)})]$, then $m = (U_1, U_2, w)$ must give

$$\mathbb{E}_{F_1}[U_1^*(c_1|m)] \geq \mathbb{E}_{F_1}[U_1^*(c_1|m')], \forall m' \in \mathcal{M}_3,$$

[16] $w^*(\cdot|\cdot)$ is defined in the previous subsection

and hence such m is not dominated by any $m' \in \mathcal{M}_3$. Indeed, since the function $(c_1, c_2) \mapsto c_1 + c_2 - \frac{1-F_2(c_2)}{f_2(c_2)}$ is increasing in both arguments by the increasing hazard rate assumption, pointwise maximization shows that w^* given by (12) and $U_2^*(\underline{c_2}) := 1 - p - \underline{c_2}$ is a solution to (16). Let $t^*(c_1, c_2) := t^*(c_2|c_1)$, for any $c_1 \in [\underline{c_1}, \overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$, where $\{t^*(\cdot|c_1)\}_{c_1 \in [\underline{c_1}, \overline{c_1}]}$ is a class of functions that implements $w^*(\cdot|c_1)$ with $U_2^*(\underline{c_2})$ for each $c_1 \in [\underline{c_1}, \overline{c_1}]$, whose existence follows from Proposition 3 and the observations above. Observe that by definition, $w^*(c_1, c_2) = w^*(c_2|c_1)$, for any $c_1 \in [\underline{c_1}, \overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$. Then for any $c_1, c_1' \in [\underline{c_1}, \overline{c_1}]$

$$
\begin{aligned}
&\mathbb{E}_{F_2}[(1 - w^*(c_1, c_2))t^*(c_1, c_2) + w^*(c_1, c_2)(p - c_1)] \\
=&\mathbb{E}_{F_2}[(1 - w^*(c_2|c_1))t^*(c_2|c_1) + w^*(c_2|c_1)(p - c_1)] \\
=&U_{1,m^*(c_1),c_1} \\
\geq&U_{1,m^*(c_1'),c_1} \\
=&\mathbb{E}_{F_2}[(1 - w^*(c_2|c_1'))t^*(c_2|c_1') + w^*(c_2|c_1')(p - c_1)] \\
=&\mathbb{E}_{F_2}[(1 - w^*(c_1', c_2))t^*(c_1', c_2) + w^*(c_1', c_2)(p - c_1)],
\end{aligned}
$$

where the first and the last equality follow from definitions, the second and the second last follow from that $t^*(\cdot|c_1)$ implements $w^*(\cdot|c_1)$ for all $c_1 \in [\underline{c_1}, \overline{c_1}]$, and the inequality follows from the optimality of $m^*(c_1)$ by (8) in the proof of Proposition 2. Also, from Proposition 2, for any $c_1 \in [\underline{c_1}, \overline{c_1}]$,

$$
\begin{aligned}
&\mathbb{E}_{F_2}[(1 - w^*(c_1, c_2))t^*(c_1, c_2) + w^*(c_1, c_2)(p - c_1)] \\
=&\mathbb{E}_{F_2}[(1 - w^*(c_2|c_1))t^*(c_2|c_1) + w^*(c_2|c_1)(p - c_1)] \\
=&U_{1,m^*(c_1),c_1} \\
\geq&p - c_1.
\end{aligned}
$$

Therefore, (w^*, t^*) is incentive compatible and individually rational for 1. On the other hand, for any $c_1 \in [\underline{c_1}, \overline{c_1}]$, $c_2', c_2 \in [\underline{c_2}, \overline{c_2}]$, since $t^*(\cdot|c_1)$ implements $w^*(\cdot|c_1)$,

$$
\begin{aligned}
&(1 - w^*(c_1, c_2))(1 - t^*(c_1, c_2)) + w^*(c_1, c_2)(1 - p - c_2) \\
=&(1 - w^*(c_2|c_1))(1 - t^*(c_2|c_1)) + w^*(c_2|c_1)(1 - p - c_2) \\
\geq&(1 - w^*(c_2'|c_1))(1 - t^*(c_2'|c_1)) + w^*(c_2'|c_1)(1 - p - c_2) \\
=&(1 - w^*(c_1, c_2'))(1 - t^*(c_1, c_2')) + w^*(c_1, c_2')(1 - p - c_2),
\end{aligned}
$$

and

$$
\begin{aligned}
&(1-w^*(c_1,c_2))(1-t^*(c_1,c_2))+w^*(c_1,c_2)(1-p-c_2) \\
=&(1-w^*(c_2|c_1))(1-t^*(c_2|c_1))+w^*(c_2|c_1)(1-p-c_2) \\
\geq&1-p-c_2.
\end{aligned}
$$

Thus, (w^*,t^*) is ex-post incentive compatible and ex-post individually rational for 2 and hence is also incentive compatible and individually rational for 2. Together, we have that (w^*,t^*) is incentive feasible and moreover, (w^*,t^*) is safe. From the previous observation, since w^* is implemented by t^* with $U_2^*(\underline{c_2})$, (w^*,t^*) is undominated among the incentive feasible protocols. This shows that (w^*,t^*) is a strong solution.

Now suppose that $m=(w,t)$ is any strong solution, by Proposition 1 we have $U_1^*(\cdot|m) \equiv U_1^*(\cdot|m^*)$. Since both m and m^* are incentive feasible, Lemma 3 then gives

$$
\begin{aligned}
U_1^*(c_1|m) &= U_1^*(\underline{c_1}|m) - \int_{\underline{c_1}}^{c_1} \bar{w}_1(x)dx, \\
U_1^*(c_1|m^*) &= U_1^*(\underline{c_1}|m^*) - \int_{\underline{c_1}}^{c_1} \bar{w}_1^*(x)dx,
\end{aligned}
$$

for all $c_1 \in [\underline{c_1}, \overline{c_1}]$. Together with $U_1^*(\cdot|m) \equiv U_1^*(\cdot|m^*)$, we then have $\bar{w}_1 \equiv \bar{w}_1^*$ Lebesgue almost everywhere. Finally, since F_1 is absolutely continuous with respect to the Lebesgue measure, $\bar{w}_1 \equiv \bar{w}_1^*$ F_1-almost everywhere as well. Therefore,

$$
\mathbb{E}[w(c_1,c_2)] = \mathbb{E}_{F_1}[\bar{w}_1(c_1)] = \mathbb{E}_{F_1}[\bar{w}_1^*(c_1)] = \mathbb{E}[w^*(c_1,c_2)] = \mathbb{E}[w^*(c_2|c_1)].
$$

This completes the proof. ■

The significance of Proposition 4 is that, even if both parties have private information, provided that player 1 has full bargaining power, ex-ante probability of breakdown would be the same as if her private information were common knowledge, which is also given by $\mathbb{E}[w^*(c_2|c_1)]$. Therefore, if some extra private information is introduced to the party with full bargaining power, there would be no additional efficiency loss incurred to the bargaining as a whole. Distortion caused by incomplete information attributes only to the party without bargaining power. To better understand the intuition behind this, let us consider again the essential trade-off faced by the principal. As remarked before, for the principal and a given type, to balance the trade-offs between optimality and information revelation, we can consider an optimization problem that reconciles potential conflicts of interests between different types by using certain weights. The proof of Proposition 4 shows that, when such weight reflects the subordinate's current information, namely, when the weight is the density f_1, the optimal result under such reconciliation will be consistent with the subordinate's information and incentives, i.e. it will be safe. If we further, as in the literature of Bayesian mechanism design,[17] regard the term $\frac{1-F_1(c_1)}{f_1(c_1)}$ as the distortion caused by private information, or in other words, the *information rent*, we can then see that, by taking f_1 as the weight, the information rent term for 1 is cancelled from (14) to (15). Specifically, if we interpret (14) separately, the first two terms captures to conflict of interests between two parties in a crisis bargaining, which is reflected by the fact that surplus is divided between the two. The third term reflects the common interest, with paying respect to information rents of both parties, since breakdown is costly to both. The last term stands for the type-specific interest, which can be think of as the loss due to different cost under incentive compatibility. When importance of different types of the principal is weighed by the information that is publicly known, the last term will then be exactly the negative of the principal's information rent and hence these two terms are cancelled out when moving from (14) to (15). More intuitively, weighing different types of the principal by the publicly known information achieves two goals. On one hand, since it reflects the subordinate's belief, the optimal solution under this weight is consistent with his available information and incentives. On the other hand, by the same reason, since the weight exactly reflects the subordinate's belief, the principal can no longer retain any information rents under such weights as if her reconciled interest is now as if it were publicly known. The idea of not revealing nor concealing any information and present it according to the public information during the selection phase, is exactly the *principle of inscrutability* (Myerson, 1983) in the mechanism design problem of an informed principal. By the principle of inscrutability, selection of protocols cannot take any advantage of private information. Furthermore, in order to retain consistency with the subordinate's available information and incentives, the selected protocol can neither take advantage of private information during the bargaining. Together, there will be no information rents to the principal, which then leads to the equivalence between cases whether the principal has private information or not.

There are two major implications from the results of this section. First, comparing the first environment and the rest, we can see that when the party of full bargaining power also has full information, that is, when the subordinate does not have private information, in any strong solution, probability of breakdown is almost always zero. In particular, ex-ante probability of breakdown is zero. This implies that, even when there is incomplete information, efficiency can still be achieved,

[17]See, for instance, Myerson (1981).

as long as this private information is on the party with full bargaining power. On the other hand, when the party with full bargaining power does not have full information, that is, when the subordinate has some private information, provided that costs are not too high or that the subordinate's privately known cost is not to "condensed" to the lowest possible ones ex-ante,[18] ex-ante probability of breakdown will be positive and thus there will be efficiency losses due to such incomplete information. Second, as remarked above, introducing any additional private information to the principal will not induce any further efficiency losses. Specifically, when the subordinate does not have private information, probability of breakdowns will still be zero and efficiency can still be achieved after introducing private information to the principal. Furthermore, even when the subordinate has some private information, efficiency loss will be the same regardless of whether the principal has private information or not. In brief, under private value environments, although it is correct to say that incomplete information may cause efficiency loss, it is not incomplete information *alone* that causes such distortions. In fact, it is the alignment between information structure and allocation of bargaining power that plays the key role. Bargaining will be more efficient if the underlying information structure is well-aligned with the allocation of bargaining power. More specifically, private information causes distortions only when it is on the "wrong side", i.e. only when it is on the subordinate.

It is also noteworthy that all these features are essentially the same as the one exhibited by the motivating example. Moreover, as we can see in Propositions 2 to 4, under a more general framework in which all possible protocols are considered by the party with full bargaining power, the outcomes are still equivalent to the simple take-it-or-leave-it bargaining protocol. That is, the take-it-or-leave-it bargaining protocol in fact attains the optimal for the principal and can be used to implement strong solutions. As noted in the introduction, it is often difficult to model crisis bargaining in international relations since there is often no obvious protocols in there existence *a priori*, nor is there any authority that can effectively enforce any protocols. Modeling international crisis bargaining requires specifications of the bargaining protocols, which sometimes requires strong assumptions. Moreover, it has also been shown that some results are sensitive even to some substantively irrelevant details. The equivalence result here further implies that, in the case when one party has full bargaining power and the uncertainties are private values, one would not need to be concerned with various specifications of bargaining protocols. As we have shown, the simple take-it-or-leave-it protocol would be enough to capture all the outcomes. This further justifies

[18] Formally, provided that $\underline{c_1} + \underline{c_2} \leq \frac{1}{f_2(\underline{c_2})}$.

Fearon's (1995) explanations of inefficient breakdowns and the *risk-return trade-off* under the case of full bargaining power and private values.

Thus far, the discussions are under the private value environments, which plays an essential roles for the undominated protocols to be safe, since the payoff of the subordinate when bargaining breaks down does not depend on the principal's type and thus ex-post individual rationality constraint is easier to be satisfied. This feature will no longer be presented under common value environments and thus the results should be re-examined thoroughly. In the next section, we will then turn to study the common value case.

4 Crisis Bargaining with Common Values

We now turn to study the environment of common value. Specifically, we now suppose that the probability of player 1 winning in a war, p is uncertain for some party. The goal is to derive the counterparts of the results in private value environment under proper solution concepts and examine if the features above still exist.

4.1 Private Information on the Principal

First, we consider the case in which only the principal has private information on the value of p. Formally, suppose that player 1 has full bargaining power, that $c_1 > 0, c_2 > 0$ are common knowledge.[19] Also, suppose that player 2's signal θ_2 is publicly observable. As in the previous section, we may without loss of generality and consider $p \sim F_1$, where $F_1(p) := \mathbb{P}_{G_1}(p(\theta_1, \theta_2) \leq p)$, for any $p \in [0,1]$ and suppose that only 1 can observe the realizations of p. Finally, we will also impose some properties on F_1, which are summarized below.

Assumption 4. *F_1 admits density f_1 and has full support on [0,1]. Moreover, the function $p \mapsto \frac{F_1(p)}{f_1(p)}$ is increasing on [0,1]. In addition, $1 - \mathbb{E}_{F_1}[p] - c_2 > 0$.*

In this environment, a protocol is then $(w, t) : [0,1] \to [0,1]^2$. Given a protocol (w, t), a realization p and player 1's report p', ex-post payoffs are respectively:

$$(1 - w(p'))t(p') + w(p')(p - c_1)$$

$$(1 - w(p'))(1 - t(p')) + w(p')(1 - p - c_2).$$

[19] That is, suppose that c_1 and c_2 are constant functions.

Therefore, from the definitions in section 2, a protocol m is incentive feasible if

$$U_1^*(p|m) \geq U_1(p', p|m), \forall p', p \in [0,1],$$
$$U_1^*(p|m) \geq p - c_1, \forall p \in [0,1],$$
$$U_{2,m}^* \geq 1 - \mathbb{E}_{F_1}[p] - c_2.$$

Similarly, summing up the expected payoffs yields the recourse constraint

$$\mathbb{E}_{F_1}[U_1^*(p|m)] + U_{2,m}^* = 1 - \mathbb{E}_{F_1}[w(p)(c_1 + c_2)]$$

As before, by the envelope arguments, the recourse constraint and the boundedness of t, the following lemma characterizes the set of incentive feasible protocols.

Lemma 4. *Suppose that $p \sim F_1$ with density f_1 and only player 1 can observe its realization. Given w, there exists t such that (w, t) is incentive feasible if and only if there exists $U_1^*(1), U_2^*$ such that:*

1. *w is non-decreasing.*
2. *$U_1^*(1) - \int_p^1 w(x)dx \geq p - c_1, \forall p \in [0,1]$.*
3. *$U_2^* \geq 1 - \mathbb{E}_{F_1}[p] - c_2$.*
4. *$U_1^*(1) + U_2^* = 1 - \mathbb{E}_{F_1}[w(p)(c_1 + c_2 - \frac{F(p)}{f(p)})]$.*
5. *$w(1)(1 - c_1) \leq U_1^*(1) \leq (1 - w(1)) + w(1)(1 - c_1)$.*

As before, conditions 1 to 3 corresponds to envelope characterizations and individual rationality constraints, except that under common value environment, the right hand side of condition 3 also involves player 2's information. Condition 4 is from the recourse constraint and Fubini's theorem, which allows us to identify elements satisfying these constraints by one constant and a function. Condition 5 follows from boundedness of t. Let $\mathcal{M}_4$ be the set of vectors $(U_1^*(1), U_2*, w(\cdot))$ in the normed linear space $\mathbb{R}^2 \times L^2([0,1])$ that satisfies the conditions in Lemma 4. Again, $\mathcal{M}_4$ is strongly closed and convex and hence is weakly compact. In fact, since for any $(U_1^*(1), U_2^*, w) \in \mathcal{M}_4$ $w : [0,1] \to [0,1]$ is non-decreasing. By the Bolzano-Weierstrass theorem, the Helley selection theorem and the Lebesgue dominant convergence theorem, $\mathcal{M}_4$ is in fact strongly compact (in L^2 norm).

Notice that in any safe protocol, condition 3 in Lemma 4 must be slack by the full support assumption and ex-post individual rationality. Therefore, by reducing player 2's payoff with a small level and increase player 1's constant term for the same level, one can easily construct an other incentive feasible protocol that dominates the original one. Thus, strong solution does not exist in this environment and such difficulties arises exactly because of the right hand side on condition 3 as remarked above. Therefore, we will be focusing on the *core* instead, which we denote as $\mathcal{C}$ henceforth.

To begin with, we first introduce another lemma, which will be useful for analyzing the core.

Lemma 5. *Define*

$$V(p) := \max_{m \in \mathcal{M}_4} U_1^*(p|m), \forall p \in [0,1].$$

Then

$$V(1) = \max\{\mathbb{E}_{F_1}[p] + c_2, 1 - c_1\}.$$

The characterizations in Lemma 4 imply that under any incentive feasible protocol m, $U_1^*(\cdot|m)$ is non-decreasing and convex. On the other hand, the preceding lemma shows that for player 1's type $p = 1$, the highest possible attainable payoff level among incentive feasible protocols is $\max\{\mathbb{E}_{F_1}[p] + c_2, 1 - c_1\}$. It is useful to describe the core by looking directly at the induced payoffs $U_1^*(\cdot|m)$, as such convex functions can recover an essentially unique incentive feasible protocol by Lemma 4. When $\mathbb{E}_{F_1}[p] + c_2 \geq 1 - c_1$, it is relatively clear that the core is a singleton $\{\overline{m}\}$, where $\overline{m} := (\mathbb{E}_{F_1}[p] + c_2, 1 - \mathbb{E}_{F_1}[p] - c_2, \mathbf{0})$. Indeed, $\overline{m}$ is incentive feasible by Lemma 4. Lemma 5 further implies that any other incentive protocol $m \in \mathcal{M}_4$ cannot give $U_1^*(1|m) > U_1^*(1|\overline{m})$. By monotonicity of any incentive feasible protocol, there is no incentive protocols that can block m_0 and furthermore, any incentive feasible protocol must be blocked by $\overline{m}$.

However, it is more difficult to describe the core when $1 - c_1 > \mathbb{E}_{F_1}[p] + c_2$. We first introduce some notations. First, define p^* by

$$\frac{F_1(p^*)}{f_1(p^*)} = c_1 + c_2, \tag{17}$$

Notice that $1 - \mathbb{E}_{F_1}[p] > c_1 + c_2$ implies that $\frac{1}{f(1)} > c_1 + c_2$. Therefore, since $\frac{F_1(p)}{f_1(p)}$ is increasing and F_1 has full support, $p^* \in (0,1)$ is uniquely well defined. Now define $\hat{p}$ by

$$\hat{p} - c_1 = \mathbb{E}_{F_1}[p | p \leq \hat{p}] + c_2. \tag{18}$$

Since $\frac{F_1(p_0)}{f_1(p_0)}$ is increasing in p_0, the function

$$p_0 \mapsto (p_0 - c_1 - c_2)F_1(p_0) - \int_0^{p_0} p dF_1(p)$$

takes value 0 at $p_0 = 0$, value $1-(c_1+c_2)-\mathbb{E}_{F_1}[p] > 0$ at $p_0 = 1$, is decreasing on $[0, p^*)$, increasing on $(p^*, 1]$ and has slope 0 at $p_0 = p^*$. Therefore, there exists a unique $\hat{p} \in (p^*, 1)$ such that

$$(\hat{p} - c_1 - c_2)F_1(\hat{p}) - \int_0^{\hat{p}} p dF_1(p) = 0.$$

Since $\hat{p} > 0$ and F_1 has full support, $\hat{p}$ uniquely solves (18). Notice that by such monotonicity and by construction, for any $p_1 \in (\hat{p}, 1]$, $p_1 - c_1 > \mathbb{E}_{F_1}[p|p \leq p_1] + c_2$ and for any $p_0 \in [0, \hat{p})$, $p_0 - c_1 < \mathbb{E}_{F_1}[p|p \leq p_0] + c_2$. For the expositions later, let

$$\hat{w}(p) := \begin{cases} 1, & \text{if } p > \hat{p} \\ 0, & \text{otherwise.} \end{cases}.$$

Let $U_1^*(1|\hat{m}) := 1 - c_1$, $U_{2,\hat{m}}^* := c_1 - \mathbb{E}_{F_1}[\hat{w}(p)(c_1 + c_2 - \frac{F_1(p)}{f_1(p)})]$ and let $\hat{m} = (U_1^*(1|\hat{m}), U_{2,\hat{m}}^*, \hat{w})$. By construction, $\hat{m} \in \mathcal{M}_4$. Moreover, $U_{2,\hat{m}}^* = 1 - \mathbb{E}_{F_1}[p] - c_2$. As remarked above, by Lemma 4, to study the elements in $\mathcal{M}_4$ we can instead examine the induced interim payoffs $U_1^*(\cdot|m)$ which are increasing and convex. The following figure illustrates such representations.

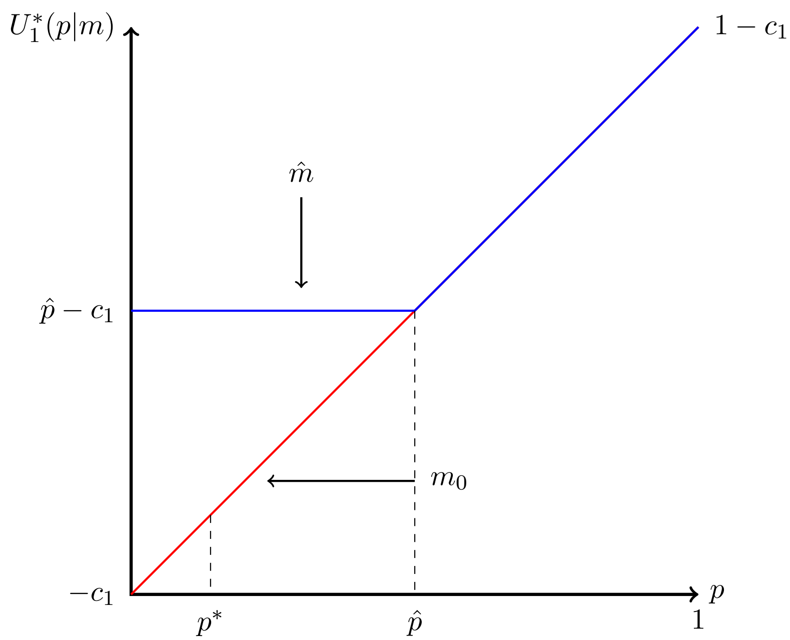

Figure 2.

Furthermore, again since $\frac{F_1(p_0)}{f_1(p_0)}$ is increasing in p_0, the function

$$p_0 \mapsto \int_{p_0}^1 (F_1(p) - (c_1 + c_2)f(p))dp$$

takes values $1 - (\mathbb{E}_{F_1}[p] + c_1 + c_2) > 0$ at $p_0 = 0$ and at $p_0 = \hat{p}$, takes value strictly greater than $1 - (\mathbb{E}_{F_1}[p] + c_1 + c_2)$ on $(0, \hat{p})$ and takes minimum at p^*. For any $p_0 \in [0, \hat{p}]$, further define

$$\alpha(p_0) := \frac{1 - (\mathbb{E}_{F_1}[p] + c_1 + c_2)}{\int_{p_0}^{1}(F(p) - (c_1 + c_2)f(p))dp}.$$

By the observations above, $\alpha(p_0) \in (0, 1]$ on $[0, \hat{p}]$, $\alpha(p_0) < 1$ on $(0, \hat{p})$, is decreasing on $(0, p^*)$ and increasing on $(p^*, \hat{p})$. With this, for each $p_0 \in [0, \hat{p}]$, define a protocol m_{p_0} by

$$\begin{aligned} w_{p_0}(p) &:= \begin{cases} \alpha(p_0), & \text{if } p \in (p_0, 1] \\ 0, & \text{otherwise} \end{cases}, \\ U_1^*(1|m_{p_0}) &:= 1 - c_1, \\ U_{2,m_{p_0}}^* &:= 1 - U_1^*(1|m_{p_0}) - \mathbb{E}_{F_1}\left[w_{p_0}(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right]. \end{aligned} \tag{19}$$

Then by construction, $m_{p_0} \in \mathcal{M}_4$ for any $p_0 \in [0, \hat{p}]$. Furthermore, define

$$\beta(p_0) := U_1^*(1|m_{p_0}) - \int_{p_0}^{1} w_{p_0}(p)dp,$$

for each $p_0 \in [0, \hat{p}]$. It then follows from the property of α that β is decreasing on $[0, \hat{p}]$, convex on $(p^*, \hat{p})$ and concave on $[0, p^*)$ with p^* being a reflexive point. In words, the above constructions is in the following order. First, for each $p_0 \in [0, \hat{p}]$, we attempt to construct some protocol that resembles $\hat{m}$ i.e. taking value $1 - c_1$ at $p = 1$, kinks once and has slope 0 left to the kink, but kink at p_0 instead of $\hat{p}$. By construction, $\alpha(p_0)$ is the slope of the right-to-the-kink part such that the whole protocol is incentive feasible. By applying this construction to all $p_0 \in [0, \hat{p}]$, we then obtain a class of slopes $\{\alpha(p_0)\}_{p_0 \in [0,\hat{p}]}$ and hence a class of values at kinks $\{U_1^*(p_0|m_{p_0})\}_{p_0 \in [0,\hat{p}]}$. Connecting these points then gives the graph of β. Figure 3 illustrates the above constructions and objects.

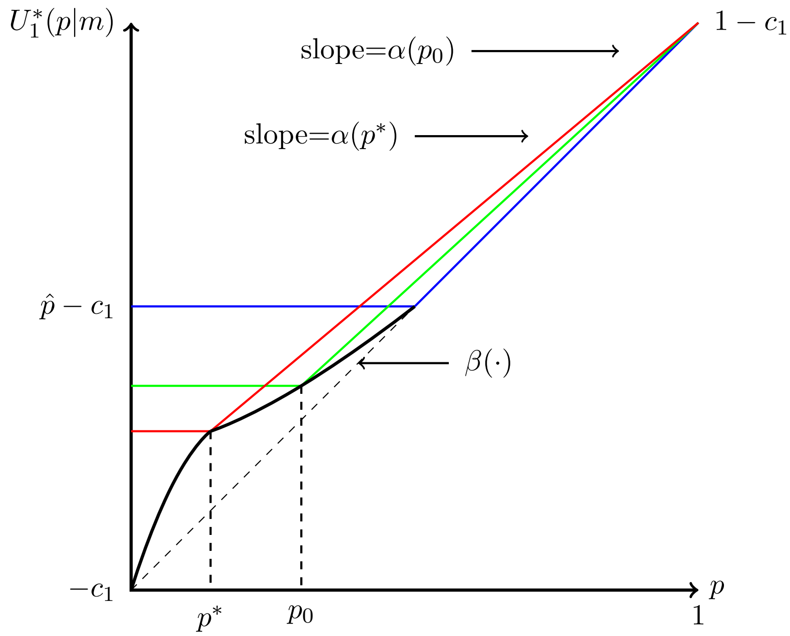

Figure 3.

Finally, define

$$\widehat{\mathcal{M}_4} := \left\{ m \in \mathcal{M}_4 \;\middle|\; \begin{array}{ll} w(p) = 0, & \forall p \in [0, p^*) \\ U_1^*(p|m) \geq \beta(p), & \forall p \in [0, \hat{p}) \\ U_1^*(1|m) = 1 - c_1 & \end{array} \right\}.$$

Figure 4 illustrates a generic element in the set $\widehat{\mathcal{M}_4}$. With these definitions, we are now able to describe some properties of the core $\mathcal{C}$.

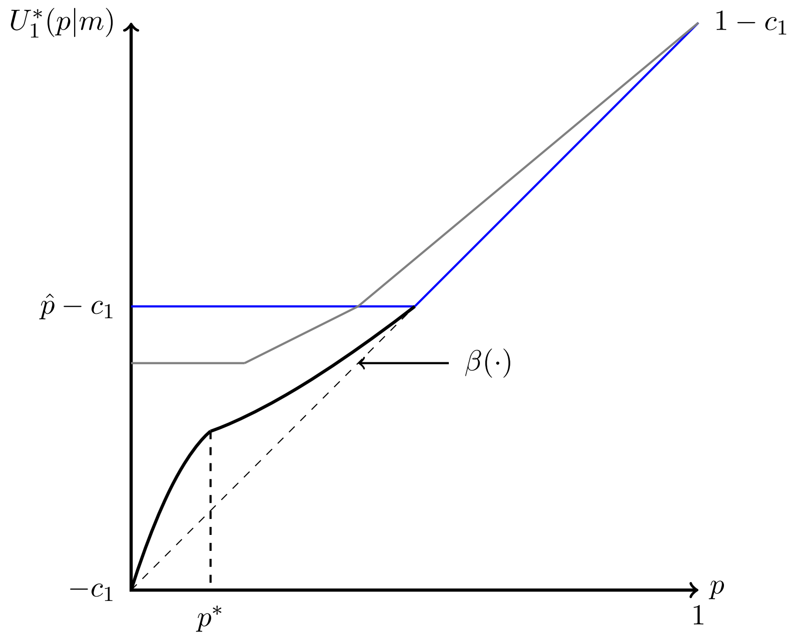

Figure 4.

Proposition 5. *Suppose that $p \sim F_1$ and only player 1 can observe its realization, that player 1 has full bargaining power, and that Assumption 4 holds. Then there is no strong solution for 1. Furthermore, if $\mathbb{E}_{F_1}[p] + c_2 \geq 1 - c_1$, then $\mathcal{C} = \{\overline{m}\}$ whereas if $\mathbb{E}_{F_1}[p] + c_2 < 1 - c_1$, $\mathcal{C} \neq \emptyset$ and $\mathcal{C} \subseteq \widehat{\mathcal{M}_4}$.*

Proof. We first show that strong solution does not exist. Indeed, let (w,t) be any safe protocol. Since m is incentive feasible, Lemma 4 implies that $m = (U_1^*(1|m), U_{2,m}^*, w) \in \mathcal{M}_4$ for some $U_1^*(1|m), U_{2,m}^*$ and hence

$$U_1^*(p|m) = U_1^*(1|m) - \int_p^1 w(x)dx \geq p - c_1,$$

for all $p \in [0,1]$. On the other hand, since m is safe,

$$U_{2,m,p}^* := (1-w(p))(1-t(p)) + w(p)(1-p-c_2) \geq 1 - p - c_2.$$

Together with

$$U_1^*(p|m) + U_{2,m,p}^* = 1 - w(p)(c_1+c_2),$$

we have

$$U_1^*(1|m) \leq p + c_2 + \int_p^1 w(x)dx - w(p)(c_1+c_2),$$

for any $p \in [0,1]$. Thus, by the full support assumption, taking expectation with respect to p under F_1 on both sides implies

$$U_1^*(1|m) < \mathbb{E}_{F_1}[p] + c_2 - \mathbb{E}_{F_1}\left[w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right]. \tag{20}$$

If $1 - w(1)c_1 \geq \mathbb{E}_{F_1}[w(p)(c_1 + c_2 - \frac{F_1(p)}{f_1(p)})]$ or $U_1^*(1|m) < 1 - w(1)c_1$, take $\epsilon > 0$ such that

$$U_1^*(1|m) + \epsilon < \max\left\{\mathbb{E}_{F_1}[p] + c_2 - \mathbb{E}_{F_1}\left[w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right], 1 - w(1)c_1\right\}. \tag{21}$$

Define $\tilde{m}$ by $(U_1^*(1|m)+\epsilon, U_{2,m}^*-\epsilon, w)$. From (21) and condition 4 in Lemma 4, $U_{2,m}^*-\epsilon > \mathbb{E}_{F_1}[p]+c_2$ and $U_1^*(1|m) + \epsilon < (1 - w(1)) + w(1)(1 - c_1)$. Therefore, by Lemma 4, $\tilde{m} \in \mathcal{M}_4$ and $U_1^*(p|\tilde{m}) > U_1^*(p|m)$, for all $p \in [0,1]$. If $1 - w(1)c_1 < \mathbb{E}_{F_1}[w(p)(c_1 + c_2 - \frac{F_1(p)}{f_1(p)})]$ and $U_1^*(1|m) = 1 - w(1)c_1$ and $w > 0$ on some set of positive Lebesgue measure, then from condition 4,

$$U_{2,m}^* = w(1)c_1 - \mathbb{E}_{F_1}\left[w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right] > 1 - \mathbb{E}_{F_1}[p] - c_2.$$

Take $\alpha \in (0,1)$ such that

$$\alpha w(1)c_1 - \mathbb{E}_{F_1}\left[\alpha w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right] > 1 - \mathbb{E}_{F_1}[p] - c_2. \tag{22}$$

Let $\tilde{m} := (U_1^*(1|m), U_{2,\tilde{m}}^*, \alpha w)$, where

$$U_{2,\tilde{m}}^* := 1 - U_1^*(1|m) - \mathbb{E}_{F_1}\left[\alpha w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right].$$

Then by (22) and Lemma 4, $\tilde{m} \in \mathcal{M}_4$ and $U_1^*(p|m) < U_1^*(p|\tilde{m})$, for all $p \in [0,1)$ since $w > 0$ with positive Lebesgue measure and $\alpha \in (0,1)$. Finally, if $w \equiv 0$ Lebesgue-almost everywhere, since w is non-decreasing, it is without loss of generality to suppose that $w(1) = 0$ since F_1 is absolute continuous and thus has zero probability on $\{1\}$, player can only be better-off by reducing $w(1)$ to 0 if $w \equiv 0$ Lebesgue almost everywhere. Then (20) reduces to $U_1^*(1|m) < \mathbb{E}_{F_1}[p] + c_2$. Together with Assumption 4, there exists $\epsilon > 0$ such that (21) holds. Therefore, $\tilde{m}$ as constructed above is in $\mathcal{M}_4$ and dominates m. Together, any safe protocol must be dominated and thus strong solution does not exist.

Suppose first that $\mathbb{E}_{F_1}[p]+c_2 \geq 1-c_1$. Let $\overline{m}$ be defined as above. Then $U_1^*(p|\overline{m}) = \mathbb{E}_{F_1}[p]+c_2$, for any $p \in [0,1]$. We claim that any incentive feasible protocol must be dominated by $\overline{m}$ and hence $\mathcal{C} = \{\overline{m}\}$. Indeed, consider any $m \in \mathcal{M}_4$, $m \neq \overline{m}$. Suppose that $\overline{m}$ does not dominate m. Then $U_1^*(p|m) > U_1^*(p|\overline{m})$ for some $p \in [0,1]$. Since $U_1^*(\cdot|\overline{m})$ is a constant function and $m \in \mathcal{M}_4$

implies that $U_1^*(\cdot|m)$ is non-decreasing. $U_1^*(1|m) > U_1^*(1|\overline{m}) = \mathbb{E}_{F_1}[p] + c_2$, contradicting to Lemma 5. Therefore, m must be dominated by $\overline{m}$.

Now suppose that $1 - c_1 > \mathbb{E}_{F_1}[p] + c_2$. Lemma 5 then implies that $U_1^*(1|m) \leq 1 - c_1$ for any $m \in \mathcal{M}_4$. We first show that $\mathcal{C} \subseteq \widehat{\mathcal{M}_4}$. Clearly, if $\widehat{\mathcal{M}_4} = \mathcal{M}_4$, then we are done. If $\widehat{\mathcal{M}_4} \subsetneq \mathcal{M}_4$. Consider any $m \in \mathcal{M}_4 \backslash \widehat{\mathcal{M}_4}$. Write $m = (U_1^*(1|m), U_{2,m}^*, w)$. Then at least one of the following must hold: (i) $w(p) > 0$ for some $p \in [0, p^*)$; (ii) $U_1^*(p|m) < \beta(p)$ for some $p \in [0, \hat{p})$; (iii) $U_1^*(1|m) < 1 - c_1$. If (i) holds, define

$$\underline{w}(p) := \begin{cases} 0, & \text{if } p \in [0, p^*] \\ w(p) & \text{if } p \in (p^*, 1] \end{cases}.$$

Since $m \in \mathcal{M}_4$ and w is non-decreasing, $w > 0$ on some $[p_0, 1] \subset [0, 1]$, and

$$1 - U_1^*(1|m) - \mathbb{E}_{F_1}\left[\underline{w}(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right] > 1 - \mathbb{E}_{F_1}[p] - c_2.$$

Take $\alpha \in (0, 1)$ such that

$$1 - U_1^*(1|m) - \mathbb{E}_{F_1}\left[\alpha\underline{w}(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right] > 1 - \mathbb{E}_{F_1}[p] - c_2. \tag{23}$$

Let $U_1^*(1|\underline{m}) := U_1^*(1|m)$, $U_{2,\underline{m}}^* := 1 - U_1^*(1|\underline{m}) - \mathbb{E}_{F_1}[\alpha\underline{w}(p)(c_1 + c_2 - \frac{F_1(p)}{f_1(p)})]$. Then for $\underline{m}$ defined as $\underline{m} := (U_1^*(1|\underline{m}), U_{2,\underline{m}}^*, \alpha\underline{w}) \in \mathcal{M}_4$ by Lemma 4 and (23). Also $U_1^*(p|\underline{m}) > U_1^*(p|m)$ for any $p \in [0, 1)$ since $w > 0$ on $[p_0, 1)$ for some $p_0 < p^*$ and $\alpha \in (0, 1)$. Together, m is blocked by $\underline{m}$ since F_1 is absolute continuous. If (ii) holds, since $U_1^*(\cdot|m)$ absolutely continuous by Lemma 4 and the fundamental theorem of calculus under Lebesgue measure,[20] there exists $0 < p_0 < p_1 \leq \hat{p}$ such that $U_1^*(p|m) < \beta(p)$ for any $p \in [p_0, p_1]$. Take any $\tilde{p} \in [p_0, p_1]$, convexity of $U_1(\cdot|m)$ and $U_1(1|m) \leq 1 - c_1$ then implies that $U_1^*(p|m) < U_1^*(p|m_{\tilde{p}})$, where $m_{\tilde{p}}$ is defined by (19). By construction, $m_{\tilde{p}} \in \mathcal{M}_4$ and hence $m_{\tilde{p}}$ blocks m since F_1 is absolute continuous. If (iii) holds and if $U_1^*(p|m) > U_1^*(p|\hat{m})$ for some $p \in [0, \hat{p}]$, then there exists some $p_1 \in (\hat{p}, 1]$ such that $U_1^*(p|\hat{m}) > U_1^*(p|m)$ for all $p \in (p_1, 1]$. Since by construction,

$$\mathbb{E}_{F_1}\left[\hat{w}(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \geq p'\right] \leq \mathbb{E}_{F_1}[p|p \geq p'] + c_2 \tag{24}$$

for any $p' \in [0, 1]$, $\hat{m}$ blocks m. Together, for any $m \in \mathcal{M}_4 \backslash \widehat{\mathcal{M}_4}$, $m \notin \mathcal{C}$. This proves the assertion.

Finally, we show that $\mathcal{C} \neq \emptyset$. For terminological conveniences, since $\mathcal{M}_4$ is isomorphic to a subset set of non-decreasing convex functions given by Lemma 4, we will index this subset by $\mathcal{M}_4$ directly, with elements m corresponding to $U_1^*(\cdot|m)$ and endow this set with the topology induced

[20] See, for reference, Royden & Fitzpatrick (2010), pp.125, Theorem 11.

by $L^2([0,1])$ norm. As remarked above, by the Helley selection theorem and the Lebesgue dominant convergence theorem, $\mathcal{M}_4$ is compact. We will invoke the following two claims, whose proofs are in the appendix.

CLAIM 1. For any $m \in \mathcal{M}_4$, if m is blocked by some $n \in \mathcal{M}_4$, then there exists $\delta > 0$ and $n' \in \mathcal{M}_4$ such that n' blocks any $m' \in N_\delta(m) \cap \mathcal{M}_4$.[21]

CLAIM 2. Let $\{n_k\}_{k=1}^K \subset \mathcal{M}_4$ be any finite collection of incentive feasible protocols. There exists $\tilde{n} \in \mathcal{M}_4$ that is not blocked by n_k for any $k \in \{1, \ldots, K\}$.

Now suppose that $\mathcal{C} = \emptyset$. Then for any $m \in \mathcal{M}_4$, there exists $n' \in \mathcal{M}_4$ such n' blocks m. By CLAIM 1, for each $m \in \mathcal{M}_4$, there exists $\delta_m > 0$, $n(m) \in \mathcal{M}_4$ such that for any $m' \in N_{\delta_m}(m) \cap \mathcal{M}_4$, m' is blocked by some $n(m) \in \mathcal{M}_4$. Since $\{N_{\delta_m}(m) | m \in \mathcal{M}_4\}$ is a family of open covers of $\mathcal{M}_4$ and $\mathcal{M}_4$ is (strongly, under the topology induced by $L^2([0,1])$ norm) compact, there exists $\{m_k\}_{k=1}^K \subset \mathcal{M}_4$, $\{\delta_1, \ldots, \delta_K\}$, $\{n_k\}_{k=1}^K \subset \mathcal{M}_4$ such that for any $m \in \mathcal{M}_4$, $m \in N_{\delta_k}(m_k) \cap \mathcal{M}_4$ for some $k \in \{1, \ldots, K\}$ and thus m is blocked by n_k for some $k \in \{1, \ldots, K\}$. However, CLAIM 2 gives that there exists $\tilde{n} \in \mathcal{M}_4$ such that $\tilde{n}$ is not blocked by any $n_j \in \{n_k\}_{k=1}^K$, a contradiction. This completes the proof. ■

Proposition 5 then yields the following corollary, which will be crucial for efficiency comparisons.

Corollary 1. *Suppose that $p \sim F_1$ and only player 1 can observe its realization, that player 1 has full bargaining power and that Assumption 4 holds. Then for any protocol (w,t) that is in the core, if $\mathbb{E}_{F_1}[p] + c_2 \geq 1 - c_1$, then $w \equiv 0$ whereas if $1 - c_1 > \mathbb{E}_{F_1}[p] + c_2$, $0 < \mathbb{E}_{F_1}[w(p)] < 1 - F_1(p^*)$*

Proof. Let (w,t) by any protocol in the core and let $m = (U_1^*(1|m), U_{2,m}^*, w)$ be its representation under Lemma 4. By Proposition 5, if $\mathbb{E}_{F_1}[p] + c_2 \geq 1 - c_1$, then $w \equiv 0$. If $1 - c_1 > \mathbb{E}_{F_1}[p] + c_2$, then $m \in \widehat{\mathcal{M}_4}$ and hence $w \equiv 0$ on $[0, p^*)$. Notice that since $\alpha(p_0) \in (0,1)$ for any $p_0 \in (0, \hat{p})$, $\beta(p_0) > p_0 - c_1$ for any $p_0 \in (0, \hat{p})$. Therefore, $U_1^*(p|m) \geq \beta(p), \forall p \in [0, \hat{p}]$ and $U_1^*(1|m) = 1 - c_1$ implies that $w < 1$ on $[p^*, p_1]$, and hence $\mathbb{E}_{F_1}[w(p)] < 1 - F_1(p^*)$. On the other hand, since $U_1^*(1|m) = 1 - c_1 > \mathbb{E}_{F_1}[p] + c_2$, if $w \equiv 0$ F_1-almost everywhere, then $U_{2,m}^* = c_1 < 1 - \mathbb{E}_{F_1}[p] - c_2$ and hence (w,t) is not individually rational for 2. Therefore, it must be that $w > 0$ with F_1 positive probability and thus $\mathbb{E}_{F_1}[w(p)] > 0$. ■

[21] As conventional, $N_\delta(m) := \{m' \in \mathcal{M}_4 | ||m' - m|| < \delta\}$ is the δ-ball around m.

The figures below illustrates the reasons why protocols not in $\widehat{\mathcal{M}_4}$ cannot be in the core.

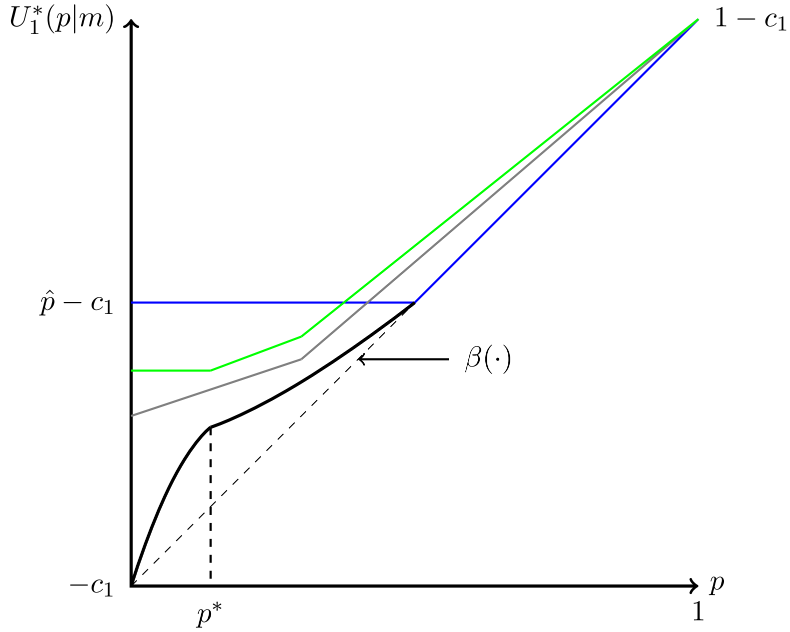

Figure 5: Case (i)

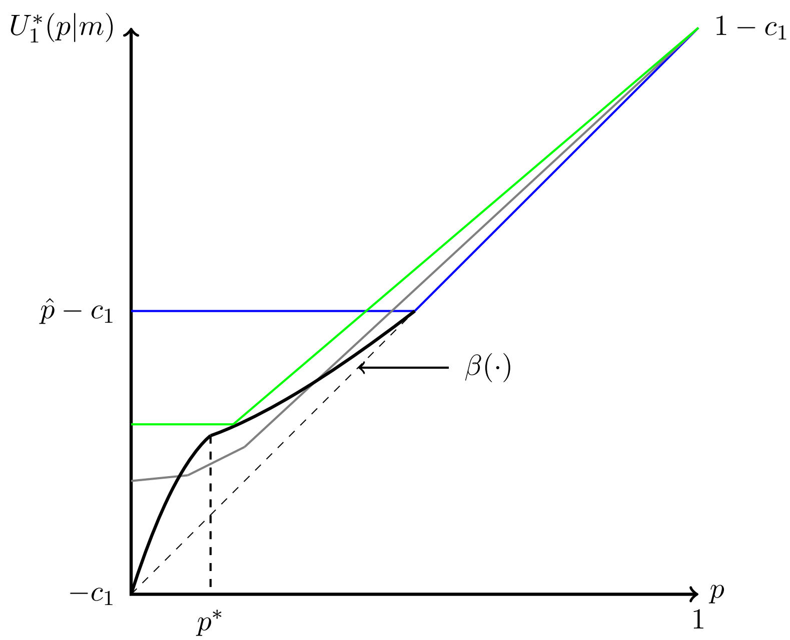

Figure 6: Case (ii)

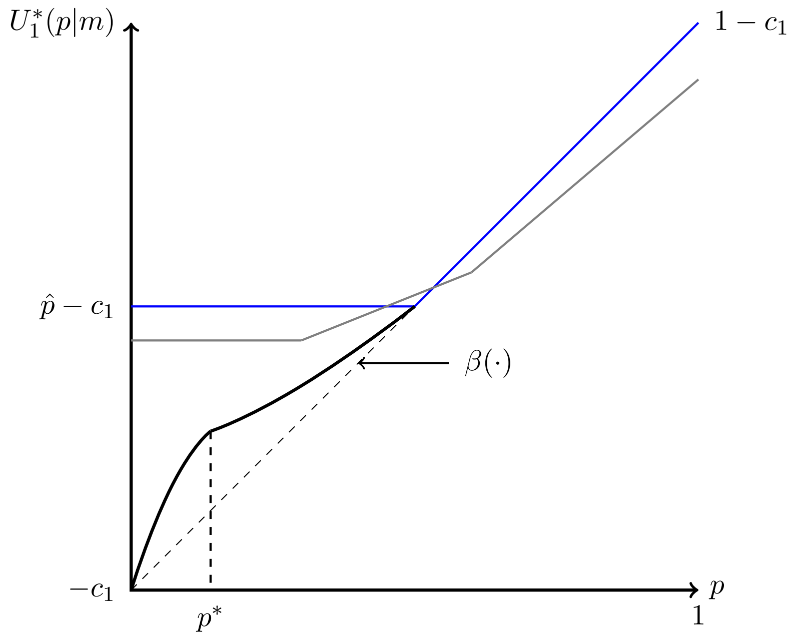

Figure 7: Case (iii)

As we can see from Proposition 5 and Corollary 1, when $\mathbb{E}_{F_1}[p] \geq 1 - c_1 - c_2$, the only possible protocol chosen by the principal, under the solution concept of core, is always peaceful and hence ex-post efficient. This corresponds to the case of private values. However, if instead, $\mathbb{E}_{F_1}[p] < 1 - c_1 - c_2$, Corollary 1 shows that there will not be any protocol in the core that always yields outcomes without breakdowns, which implies that always peaceful, and hence ex-post efficiency result, contrary to the private value case, is still not attainable even if bargaining power is assigned to the party with full information. To understand this, observe that the key difference between private value and common value environments is that in common value, one's private information will affect the other's payoff directly. As such, there is a first-order effect in misrepresenting information for the informed party—"bluffing" can not only be used to manipulate the opponent's belief on one's *resolution*, and thus willingness to pay in sharing surpluses, but also can affect the opponent's belief on how much they could expect if such bargaining breaks down, and hence the opponent's willingness to pay as well. Therefore, the incentive constraint for truthful information revelation is stronger in the common value case since the "weak" (lower p) would have even stronger incentive to mimic the "strong" (high p). As a result, whether there will be efficiency loss then depends on the relative magnitude between expected relative power and costs. In words, Corollary 1 can be think of as introducing the effect of *mis-alignment between bargaining power and distribution of* *power*. If $\mathbb{E}_{F_1}[p]$ is large enough, it means that as a common knowledge, both parties know that the principal is "strong enough" and hence the subordinate would be willing to accept more demanding protocols to her, which is rewarding enough for the principal with low realizations to outweigh the incentives to misrepresent. On the other hand, if $\mathbb{E}_{F_1}[p]$ is not large enough, it then indicates that even if the information structure is perfectly aligned with allocation of bargaining power, if the *distribution of power* is not reflected by such allocation, there would still be efficiency losses—exactly due to incentives to mis-represent. In brief, a distinct feature between private value and common value environments discovered here is that, in common value environments, besides the alignment between information structure and bargaining power, the alignment between distribution of power and bargaining power also plays a role in affecting efficiency. In the next subsection, we will then compare which of the alignments is more effective.

Before concluding this subsection, we further remark that the condition in Proposition 5 and Corollary 1, $\mathbb{E}_{F_1}[p] \geq 1-c_1-c_2$, is exactly the necessary and sufficient condition for *the existence* of always peaceful protocol in Fey & Ramsey (2011) if the private information is reduced to one-sided. By further specifying the allocation of bargaining power, the result above further sharpens theirs in a sense that it is in fact a necessary and sufficient condition for a protocol to be always peaceful *as an outcome that will emerge for sure*, not just for *existence* per se.

4.2 Private Information on the Subordinate

As remarked above, we now turn to examine the effects on efficiency when allocation of bargaining power and information structure are mis-aligned. Formally, suppose now that player 2 has full bargaining power and the information structure is unchanged. That is $p \sim F_1$ and only player 1 can observe its realization. We also retain Assumption 4. As such, we can then directly compare the outcomes in these two different assignments of bargaining powers and see the effect of mis-alignment between bargaining power and information. Since the information structure is unchanged, the set of incentive feasible protocols are unchanged and Lemma 4 still applies. Also, since player does not have private information, as in the private value environment, strong solution exists and is reduced to the simple optimization problem

$$\max_{m \in \mathcal{M}_4} U^*_{2,m}.$$

Recall that we have defined p^* as the solution to

$$\frac{F_1(p^*)}{f_1(p^*)} = (c_1 + c_2)$$

when $\mathbb{E}_{F_1}[p] + c_2 < 1 - c_1$. In this case, under Assumption 4, such $p^* \in (0,1)$ is uniquely defined since $\mathbb{E}_{F_1}[p] + c_2 < 1 - c_1$ implies $\frac{1}{f_1(1)} > c_1 + c_2$ and $\frac{F_1}{f_1}$ is increasing. We now extend this definition for a more concise description. That is, define p^* by

$$\frac{F_1(p^*)}{f_1(p^*)} = (c_1 + c_2)$$

if $\frac{1}{f_1(1)} \geq 1$ and $p^* = 1$ otherwise. Then such $p^* \in (0,1]$ is always uniquely well-defined under Assumption 4. Furthermore, define a protocol $m^* := (U_1^*(1|m^*), U_{2,m^*}^*, w^*)$ by

$$\begin{aligned} w^*(p) &:= \begin{cases} 1, & \text{if } p > p^* \\ 0, & \text{otherwise} \end{cases}, \\ U_1^*(1|m^*) &:= 1 - c_1, \\ U_{2,m^*}^* &:= 1 - U_1^*(1|m^*) - \mathbb{E}_{F_1}\left[w^*(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right]. \end{aligned} \tag{25}$$

Also, as in the previous section, given any $w : [0,1] \to [0,1]$, we say that a function $t : [0,1] \to [0,1]$ implements w if there exists $U_1^*(1), U_2^*$ such that $(U_1^*(1), U_2^*, w) \in \mathcal{M}_4$. With these definitions, we can then state the main result in this subsection.

Proposition 6. *Suppose that $p \sim F_1$ and only player 1 can observe its realization, that player 2 has full bargaining power, and that Assumption 4 holds. Then there exists $t^* : [0,1] \to [0,1]$ such that (w^*, t^*) is the (essentially unique) strong solution.*

Proof. As remarked above, solving for strong solution reduces to solving for the problem

$$\max_{m \in \mathcal{M}_4} U_{2,m}^*. \tag{26}$$

By Lemma 4, we can solve (26) by considering the following problem:

$$\begin{aligned} \max_{U_1^*(1), w(\cdot)} & \left[1 - U_1^*(1) - \mathbb{E}_{F_1}\left[w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right]\right] \\ \text{s.t. } & w \quad \text{is non-decreasing}, \\ & U_1^*(1) \geq 1 - c_1. \end{aligned} \tag{27}$$

If the solution to (27) further satisfies

$$U_1^*(1) - \int_p^1 w(x)dx \geq p - c_1, \tag{28}$$

$$w(1)(1 - c_1) \leq U_1^*(1) \leq (1 - w(1)) + w(1)(1 - c_1), \tag{29}$$

then it solves (26). By pointwise maximization, $U_1^*(1|m^*), w^*$ defined in (24) is the essentially unique solution to (26), where monotonicity of $\frac{F_1}{f_1}$ is used for monotonicity of w^*. Furthermore, if $p^* \in (0,1)$, then $w(1) = 1$ and hence (28) is satisfied. On the other hand,

$$U_1^*(1|m^*) - \int_p^1 w^*(x)dx = \begin{cases} p - c_1, & \text{if } p \in (p^*, 1] \\ p^* - c_1, & \text{if } p \in [0, p^*] \end{cases}$$

and hence (29) is satisfied. Also, if $p^* = 1$, then $w \equiv 0$ and (28), (29) are satisfied. Therefore, m^* is the essentially unique solution to problem (26). Furthermore, since $m^* \in \mathcal{M}_4$, Lemma 4 ensures that there exists $t^* : [0,1] \to [0,1]$ that implements w^*. For such t^*,

$$(1 - w^*(p))t^*(p) + w^*(p)(p - c_1) = 1 - c_1 - \int_p^1 w^*(x)dx,$$

for any $p \in [0,1]$. Since w^* is the essentially unique solution to (26), (w^*, t^*) is the essentially unique strong solution. ■

With Proposition 5 and Proposition 6, we then have the following result, which is the counterpart of the results in the private value environment

Corollary 2. *Suppose that $p \sim F_1$ and only player 1 can observe its realization and that Assumption 4 holds. Then (ex-ante) probability of breakdown is always strictly greater if player 2 has full bargaining power than if player 1 has full bargaining power if the former is positive. Furthermore, if the (ex-ante) probability of breakdown when player 2 has full bargaining power is zero, then it must also be zero when player 1 has full bargaining power.*

Proof. From Proposition 6, when player 2 has full bargaining power, the essentially unique strong solution gives ex-ante probability of breakdown as $\mathbb{E}_{F_1}[w^*(p)] = 1 - F_1(p^*)$. On the other hand, when player 1 has full bargaining power, let (w,t) be any protocol in the core, if $p^* \in (0,1)$, Corollary 1 gives that $\mathbb{E}_{F_1}[w(p)] < 1 - F_1(p^*)$. If $p^* = 1$, it must be that $\frac{1}{f_1(1)} \leq c_1 + c_2$, which further implies that $\mathbb{E}_{F_1}[p] + c_2 \geq 1 - c_1$. Corollary 1 then implies that $\mathbb{E}_{F_1}[w(p)] = 0$. ■

From the results above, in addition to the remarks in the previous subsection, we can further see that although both the alignment between *distribution of power* and allocation of bargaining power and the alignment between *information structure* and the allocation of bargaining power would affect efficiency of a crisis bargaining, the mis-alignment between *information structure* and bargaining power unambiguously causes more distortions. Indeed, as we can see from Corollary 1 and Corollary 2, if the allocation of bargaining power is well-aligned with both distribution of power and information structure, as in the private value environment, ex-post efficiency can be achieved. If it is only well-aligned with the information structure but not the distribution of power, there will be some efficiency losses. However, such losses are always milder than the those caused by mis-alignment of bargaining power and information by Corollary 2. Furthermore, although it is possible, under some specific relationships between costs and the common prior F_1,[22] that ex-post efficiency can be achieved even if information structure and allocation of bargaining power are mis-aligned, Assumption 4 ensures that such condition must occur under a more subordinate-favored distribution of power and thus if the bargaining power were shifted, it must have been well-aligned with distribution of power and therefore ex-post efficiency can still be achieved. Together, the main feature in the private value case, that bargaining outcome is more efficient when allocation of bargaining power is well-aligned with the information structure, still presents in the common value environment. On the other hand, in common value environments, the alignment between distribution of power and bargaining power also matters, but is always milder then that between information structure and bargaining power.

We will now turn to examine the case when both parties have private information on the common valued component. One of the goals of the analyses below is to study the counterpart of the second result in private value environment, that when extra private information is introduced to the party with full bargaining power, there will be no further efficiency losses.

4.3 Private Information on Both Parties

We now examine the case where both parties have private information. Formally, from the previous general framework. Suppose that the functions $c_1(\cdot), c_2(\cdot)$ are constant. Without loss of generality, normalize the signal spaces Θ_1 and Θ_2 to $[0, 1]$. We will suppose that player 1 has full bargaining power hereafter. Below, we will state some assumptions on the behavior of the function $p : [0, 1]^2 \to [0, 1]$, as well as the behaviors of the underlying CDF G_1 and G_2.

Assumption 5. *The function $p : [0, 1]^2 \to [0, 1]$ has the following property:*

1. *p is increasing in the first argument and decreasing in the second argument.*

2. *Partial derivatives, p_{θ_1}, p_{θ_2} of p exists everywhere.*

3. *p_{θ_1} is non-decreasing and p_{θ_2} is non-increasing. Moreover, p_{θ_1}, p_{θ_2} are continuous.*

[22] Namely $\frac{1}{f_1(1)} \leq c_1 + c_2$

4. $\{p_{\theta_1}(\cdot,\theta_2)|\theta_2 \in [0,1]\}$ *and* $\{p_{\theta_2}(\theta_1,\cdot)|\theta_1 \in [0,1]\}$ *are uniformly integrable.*

5. $p(0,\cdot) \equiv \underline{p}$, $p(1,\cdot) \equiv \overline{p}$; $p(\cdot,0) \equiv \overline{p}$, $p(\cdot,1) \equiv \underline{p}$ *for some* $0 \leq \underline{p} < \overline{p} \leq 1$ *with* $1-\underline{p}-c_2 > 0$ *and* $\overline{p} - c_1 > 0$.

Assumption 6. *G_1 and G_2 admit densities g_1 and g_2, respectively and both have full support on $[0,1]$. Furthermore, the functions $\theta_1 \mapsto \frac{G_1(\theta)}{g_1(\theta_1)}$ and $\theta_2 \mapsto \frac{G_2(\theta_2)}{g_2(\theta_2)}$ are increasing.*[23]

Condition 1 in Assumption 5 means that the signals θ_1 and θ_2 are ordered in a way that for each player, higher the signal is, more relatively powerful she would be.[24] Conditions 2 to 4 are some technical conditions. Condition 5 means that, for each player, whenever the "strongest" ("weakest") signal is observed, she would know for sure that her relative power is the strongest (weakest). For conciseness, write $\bar{p_1}(\theta_1) := \mathbb{E}_{G_2}[p(\theta_1,\theta_2)]$ for any $\theta_1 \in [0,1]$ and $\bar{p_2}(\theta_2) := \mathbb{E}_{G_1}[p(\theta_1,\theta_2)]$ for any $\theta_2 \in [0,1]$. Also, given w, write $v_1(\theta_1,\theta_2) := w(\theta_1,\theta_2)p_{\theta_1}(\theta_1,\theta_2)$, $v_2(\theta_1,\theta_2) := -w(\theta_1,\theta_2)p_{\theta_2}(\theta_1,\theta_2)$, for all $\theta_1,\theta_2 \in [0,1]$. Finally, let $\bar{v_1}(\theta_1) := \mathbb{E}_{G_2}[v_1(\theta_1,\theta_2)]$ for all $\theta_1 \in [0,1]$ and $\bar{v_2}(\theta_2) := \mathbb{E}_{G_1}[v_2(\theta_1,\theta_2)]$ for all $\theta_2 \in [0,1]$.

Before stating the main results, since the main goal in this subsection is to understand whether introducing additional private information to player 1 will cause extra efficiency loss, we first study the benchmark case in a fictitious environment in which realizations of θ_1 are commonly observed.

Lemma 6. *Suppose that $\theta_2 \sim G_2$ with density g_2 and only player 2 can observe its realization, that $\theta_1 \sim G_1$ with density g_1 but both players can observe its realization, and that Assumption 5 holds. Then for each $\theta_1 \in [0,1]$, given $w(\cdot|\theta_1)$, if there exists $t(\cdot|\theta_1)$ such that $(w(\cdot|\theta_1), t(\cdot|\theta_1))$ is incentive feasible, then there exists $U_1^*, U_2^*(1)$ such that*

1. $v_2(\cdot|\theta_1)$ *is non-decreasing.*

2. $U_1^* \geq \bar{p_1}(\theta_1) - c_1$

3. $U_2^*(1) - \int_{\theta_2}^1 v_2(x|\theta_1)dx \geq 1 - p(\theta_1,\theta_2) - c_2, \forall \theta_2 \in [0,1]$.

4. $U_1^* + U_2^*(1) = 1 - \mathbb{E}_{G_2}[w(\theta_2|\theta_1)(c_1 + c_2 + p_{\theta_2}(\theta_1,\theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)})]$.

5. $w(1)(1-\underline{p}-c_2) \leq U_2^*(1) \leq 1 - w(1) + w(1)(1-\underline{p}-c_2)$.

[23]Again, we remark that the monotonicity assumption for $\frac{G_1}{g_1}$ is redundant for the results in this section, but will be useful in the later section when intermediate allocation of bargaining power is considered.

[24]Since the general definition in section 2 does not have restrictions on signal spaces, this assumption is in fact without loss of generality

Conversely, if there exists $U_1^*, U_2^*(1)$ such that conditions 2 through 5 hold and $w(\cdot|\theta_1)$ is non-decreasing, then there exits $t(\cdot|\theta_1)$ such that $(w(\cdot|\theta_1), t(\cdot|\theta_1))$ is incentive feasible.

As before, for each $\theta_1 \in [0,1]$ let $\mathcal{M}_5(\theta_1)$ be the set of vectors $(U_1^*, U_2^*(1), w)$ that satisfies conditions 1 to 5 in Lemma 6. By the same reasons as above, $\mathcal{M}_5(\theta_1)$ is (weakly) compact and convex for any $\theta_1 \in [0,1]$. Given any $\theta_1 \in [0,1]$, for any $m \in \mathcal{M}_5(\theta_1)$ such that w is implementable, let U_{1,m,θ_1}^* and $U_2^*(\cdot|m,\theta_1)$ denote the equilibrium payoffs under m given θ_1. For each $\theta_1 \in [0,1]$, let $w^*(\cdot|\theta_1)$ be defined as

$$w^*(\theta_2|\theta_1) = \begin{cases} 1, & \text{if } c_1 + c_2 + p_{\theta_2}(\theta_1,\theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)} < 0 \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

Then, since by 3 in Assumption 5 and Assumption 6, the function

$$(\theta_1,\theta_2) \mapsto p_{\theta_2}(\theta_1,\theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)}$$

is increasing in both arguments and hence $w^*(\cdot|\theta_1)$ is non-decreasing for each $\theta_1 \in [0,1]$ and $w^*(\theta_2|\cdot)$ is also non-decreasing for each $\theta_2 \in [0,1]$. The following proposition is then a benchmark.

Proposition 7. *Suppose that $\theta_2 \sim G_2$ and only player 2 can observe its realization, that $\theta_1 \sim G_1$ but both players can observe its realization, that player 1 has full bargaining power, and that Assumption 5 and 6 holds. Then for each $\theta_1 \in [0,1]$, there exists $t^*(\cdot|\theta_1) : [0,1] \to [0,1]$ such that $(w^*(\cdot|\theta_1), t^*(\cdot|\theta_1))$ is the essentially unique strong solution for 1.*

Proof. For any $\theta_1 \in [0,1]$. By Lemma 6, as in the previous subsection, solving for strong solutions reduced to solving for the maximization problem

$$\max_{m\in\mathcal{M}_5(\theta_1)} U_{1,m,\theta_1}^* \text{ s.t. } w \text{ is implementable.} \tag{31}$$

Using Lemma 6, (31) can be solved by considering the problem

$$\begin{aligned} &\max_{w(\cdot).U_2^*(1)} \left[1 - U_2^*(1) - \mathbb{E}_{G_2}\left[w(\theta_2)\left(c_1 + c_2 + p_{\theta_2}(\theta_1,\theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)}\right)\right]\right] \\ &\quad \text{s.t. } U_2^*(1) \geq 1 - \underline{p} - c_2. \end{aligned} \tag{32}$$

By Lemma 6, if the solution of (32) also satisfies

$$U_2^*(1) - \int_{\theta_2}^1 v_2(x)dx \geq 1 - p(\theta_1,\theta_2) - c_2, \forall\theta_2 \in [0,1], \tag{33}$$

$$1 - U_2^*(1) - \mathbb{E}_{G_2}\left[w(\theta_2)\left(c_1 + c_2 + p_{\theta_2}(\theta_1,\theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)}\right)\right] \geq \bar{p_1}(\theta_1) - c_1, \tag{34}$$

$$w(1)(1 - \underline{p} - c_2) \leq U_2^*(1) \leq 1 - w(1) + w(1)(1 - \underline{p} - c_2), \tag{35}$$

and $w(\cdot)$ is non-decreasing ,then such solution yields a solution to (31) as well. Pointwise maximization shows that for each $\theta_1 \in [0,1]$ $(w^*(\cdot|\theta_1), 1-\underline{p}-c_2)$ is the (essentially) unique solution to (32). Moreover, as observed above, $w^*(\cdot|\theta_1)$ is indeed non-decreasing. We now verify that for $U_2^*(1) = 1-\underline{p}-c_2$, (33)-(35) are satisfied. Indeed, for (33), for any $\theta_2 \in [0,1]$,

$$\begin{aligned}
&1-\underline{p}-c_2+\int_{\theta_2}^1 p_{\theta_2}(\theta_1,x)w^*(x|\theta_1)dx \\
\geq& 1-\underline{p}-c_2+\int_{\theta_2}^1 p_{\theta_2}(\theta_1,x)dx \\
=& 1-\underline{p}-c_2+\underline{p}-p(\theta_1,\theta_2) \\
=& 1-p(\theta_1,\theta_2)-c_2,
\end{aligned}$$

where the first inequality follows from $w^*(\cdot|\theta_1) \leq 1$ and $p_{\theta_2} < 0$ and the equality below follows from differentiability of $p(\theta_1,\cdot)$ and the fundamental theorem of calculus. For (34),

$$\begin{aligned}
&\underline{p}+c_2-\mathbb{E}_{G_2}\left[w^*(\theta_2|\theta_1)\left(c_1+c_2+p_{\theta_2}(\theta_1,\theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)}\right)\right] \\
\geq& \underline{p}+c_2-\mathbb{E}_{G_2}\left[c_1+c_2+p_{\theta_2}(\theta_1,\theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)}\right] \\
=& \underline{p}-c_1-\int_0^1 p_{\theta_2}(\theta_1,\theta_2)G_2(\theta_2)d\theta_2 \\
\geq& \underline{p}-c_1-\int_0^1 p_{\theta_2}(\theta_1,\theta_2)d\theta_2 \\
=& \overline{p}-c_1 \\
\geq& \bar{p_1}(\theta_1)-c_1,
\end{aligned}$$

where the first inequality follows from optimality of $w^*(\cdot|\theta_1)$ in (32), the third equality comes from $p_{\theta_2} < 0$ and $G_2 \leq 1$ and the equality below is from the fundamental theorem of calculus. Finally, for (35), if $w^*(1|\theta_1) = 1$, then (34) holds in equality on both sides. If $w^*(1|\theta_1) = 0$, then (34) holds by the assumption $1-\underline{p}-c_2 > 0$. Therefore, let

$$\begin{aligned}
&U_2^*(1|m^*(\theta_1),\theta_1) := 1-\underline{p}-c_2 \\
&U_{1,m^*(\theta_1),\theta_1}^* := \underline{p}+c_2-\mathbb{E}_{G_2}\left[w^*(\theta_2|\theta_1)\left(c_1+c_2+p_{\theta_2}(\theta_1,\theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)}\right)\right]
\end{aligned}$$

and let $m^*(\theta_1) := (U_{1,m^*(\theta_1),\theta_1}^*, U_2^*(1|m^*(\theta_1),\theta_1), w^*(\cdot|\theta_1))$. Then $m^*(\theta_1)$ is the essentially unique solution to (31) for any fixed θ_1. Since $w^*(\cdot|\theta_1)$ is non0decreasing for each $\theta_1 \in [0,1]$, by Lemma 6, we can take the function $t^*(\cdot|\theta_1)$ that implements $w^*(\cdot|\theta_1)$ with $1-\underline{p}-c_2$ for each $\theta_1 \in [0,1]$.

Then, since $w^*(\cdot|\theta_1)$ is essentially unique, and since for any $\theta_2 \in [0,1]$,

$$(1-w^*(\theta_2|\theta_1))(1-t^*(\theta_2|\theta_1)) + w^*(\theta_2|\theta_1)(1-p(\theta_1,\theta_2)-c_2) = 1-\underline{p}-c_2+\int_{\theta_2}^1 p_{\theta_2}(\theta_1,x)w^*(x|\theta_1)dx,$$

$(w^*(\cdot|\theta_1), t^*(\cdot|\theta_1))$ is the essentially unique strong solution. ■

In fact, Proposition 7 is a natural extension of Proposition 6 in which although player 1's signal is publicly observed, it is still considered as a parameter and the strong solution would be a function of realization of θ_1. Notice that by optimality of problem (32) in the proof of Proposition 7, for any $\theta_1', \theta_1 \in [0,1]$, $U^*_{1,m^*(\theta_1),\theta_1} \geq U^*_{1,m^*(\theta_1'),\theta_1}$. This also characterizes the ex-ante probability of breakdown when θ_1 is commonly observed. Such probability is given by $\mathbb{E}[w^*(\theta_2|\theta_1)] = \mathbb{P}(\{c_1+c_2+p_{\theta_2}(\theta_1,\theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)} < 0\})$, which is positive whenever $c_1+c_2 < -p_{\theta_2}(1,1)\frac{1}{g_2(1)}$. With the benchmark being developed, we can now begin the analyses when realizations of θ_1 is privately observed by player 1. The following lemma gives the characterization of incentive feasible protocols under this environment.

Lemma 7. *Suppose that $\theta_1 \sim G_1$ with densities g_1 and g_2, respectively, $\theta_2 \sim G_2$, that only player 1 can observe realizations of θ_1 and only player 2 can observe realizations of θ_2, and that Assumption 5 holds. Then given w, if there exists t such that (w,t) is incentive feasible, then there exists $U_1^*(1), U_2^*(1)$ such that:*

1. *$\bar{v}_1, \bar{v}_2$ are non-decreasing.*

2. *$U_1^*(1) - \int_{\theta_1}^1 \bar{v}_1(x)dx \geq \bar{p}_1(\theta_1) - c_1$, $\forall \theta_1 \in [0,1]$*

3. *$U_2^*(1) - \int_{\theta_2}^1 \bar{v}_2(x)dx \geq 1 - \bar{p}_2(\theta_2) - c_2, \forall \theta_2 \in [0,1]$.*

4. *$U_1^*(1) + U_2^*(1) = 1 - \mathbb{E}[w(\theta_1,\theta_2)(c_1+c_2+p_{\theta_2}(\theta_1,\theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)} - p_{\theta_1}(\theta_1,\theta_2)\frac{G_1(\theta_1)}{g_1(\theta_1)})]$.*

5. *$\bar{w}_1(1)(\overline{p}-c_1) \leq U_1^*(1) \leq 1 - \bar{w}_1(1) + \bar{w}_1(1)(\overline{p}-c_1)$*

6. *$\bar{w}_2(1)(1-\underline{p}-c_2) \leq U_2^*(1) \leq 1 - \bar{w}_2(1). + \bar{w}_2(1)(1-\underline{p}-c_2)$.*

Conversely, if there exists $U_1^*(1), U_2^*(1)$ such that conditions 2 through 6 hold and w is non-decreasing in both arguments, then there exits t such that (w,t) is incentive feasible,

Let $\mathcal{M}_5$ denote the set of vectors $(U_1^*(1), U_2^*(1), w(\cdot,\cdot))$ satisfying conditions 1 to 6 in Lemma 7. Again, $\mathcal{M}_5$ is (weakly) compact and convex. Using this lemma and the benchmark above, we are then able to obtain the main result in this subsection.

Proposition 8. *Suppose that $\theta_1 \sim G_1$, $\theta_2 \sim G_2$, that only player 1 can observe realizations of θ_1 and only player 2 can observe realizations of θ_2, that player 1 has full bargaining power, and that Assumption 5 and 6 hold. Then there exists a strong solution (w^*, t^*) in which $\mathbb{E}[w^*(\theta_1, \theta_2)] = \mathbb{E}[w^*(\theta_2|\theta_1)]$.*

Proof. Consider the problem

$$\max_{m \in \mathcal{M}_5} \mathbb{E}_{G_1}[U_1^*(\theta_1|m)] \text{ s.t. } w \text{ is (interim) implementable .} \tag{36}$$

By the full support Assumption and Lemma 7, any solution of (36) must be undominated among all the incentive feasible protocols. Using Lemma 7, we can solve (36) by considering

$$\begin{aligned} &\max_{U_2^*(1), w(\cdot,\cdot)} \left[1 - U_2^*(1) - \mathbb{E}\left[w(\theta_1, \theta_2)\left(c_1 + c_2 + p_{\theta_2}(\theta_1, \theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)}\right)\right]\right] \\ &\quad \text{s.t. } U_2^*(1) \geq 1 - \underline{p} - c_2. \end{aligned} \tag{37}$$

If a solution to (37) can be implemented by some function $t : [0,1]^2 \to [0,1]$, then it must be a solution to (36) and hence gives an undominated protocol. Define the function $w^* : [0,1]^2 \to [0,1]$ by

$$w^*(\theta_1, \theta_2) := w^*(\theta_2|\theta_1), \forall \theta_1, \theta_2 \in [0,1] \tag{38}$$

and let $U_2^*(1|m^*) := 1 - \underline{p} - c_2$. Then $(U_2^*(1|m^*), w^*)$ is a solution to (37). Notice that by construction, w^* is indeed non-decreasing in both arguments. By Proposition 7 and the observations above, for each $\theta_1 \in [0,1]$, $t^*(\cdot|\theta_1)$ implements $w^*(\cdot|\theta_1)$ with $1 - \underline{p} - c_2$. Let $t^*(\theta_1, \theta_2) := t^*(\theta_2|\theta_1)$ for all $\theta_1, \theta_2 \in [0,1]$. We now claim that t^* implements w^* and, moreover, (w^*, t^*) is safe. Indeed, for 2, since $t^*(\cdot|\theta_1)$ implements $w^*(\cdot|\theta_1)$ with $1 - \underline{p} - c_2$ for each $\theta_1 \in [0,1]$.

$$\begin{aligned} &(1 - w^*(\theta_1, \theta_2))(1 - t^*(\theta_1, \theta_2)) + w^*(\theta_1, \theta_2)(1 - p(\theta_1, \theta_2) - c_2) \\ =&(1 - w^*(\theta_2|\theta_1))(1 - t^*(\theta_2|\theta_1)) + w^*(\theta_2|\theta_1)(1 - p(\theta_1, \theta_2) - c_2) \\ \geq&(1 - w^*(\theta_2'|\theta_1))(1 - t^*(\theta_2'|\theta_1)) + w^*(\theta_2'|\theta_1)(1 - p(\theta_1, \theta_2) - c_2) \\ =&(1 - w^*(\theta_1, \theta_2'))(1 - t^*(\theta_1, \theta_2')) + w^*(\theta_1, \theta_2')(1 - p(\theta_1, \theta_2) - c_2), \end{aligned}$$

for any $\theta_2', \theta_2 \in [0,1]$ and hence (w^*, t^*) is ex-post incentive compatible for 2. Also,

$$\begin{aligned} &(1 - w^*(\theta_1, \theta_2))(1 - t^*(\theta_1, \theta_2)) + w^*(\theta_1, \theta_2)(1 - p(\theta_1, \theta_2) - c_2) \\ =&(1 - w^*(\theta_2|\theta_1))(1 - t^*(\theta_2|\theta_1)) + w^*(\theta_2|\theta_1)(1 - p(\theta_1, \theta_2) - c_2) \\ \geq&1 - p(\theta_1, \theta_2) - c_2, \end{aligned}$$

for any $\theta_2 \in [0,1]$ and hence (w^*, t^*) is ex-post individually rational for 2. In particular, (w^*, t^*) is incentive compatible and individually rational for 2. On the other hand, for 1, from optimality of $(1-\underline{p}-c_2, w^*(\cdot|\theta_1))$ in problem (32) and the fact that $t^*(\cdot|\theta_1)$ implements $w^*(\cdot|\theta_1)$ with $1-\underline{p}-c_2$, for any $\theta_1', \theta_1 \in [0,1]$,

$$\begin{aligned}
&\mathbb{E}_{G_2}[(1-w^*(\theta_1,\theta_2))t^*(\theta_1,\theta_2)+w^*(\theta_1,\theta_2)(p(\theta_1,\theta_2)-c_1)]\\
=&\mathbb{E}_{G_2}[(1-w^*(\theta_2|\theta_1))t^*(\theta_2|\theta_1)+w^*(\theta_2|\theta_1)(p(\theta_1,\theta_2)-c_1)]\\
=&U^*_{1,m^*(\theta_1),\theta_1}\\
\geq&U_{1,m^*(\theta_1'),\theta_1}\\
=&\mathbb{E}_{G_2}[(1-w^*(\theta_2|\theta_1'))t^*(\theta_2|\theta_1')+w^*(\theta_2\theta_1')(p(\theta_1,\theta_2)-c_1)]\\
=&\mathbb{E}_{G_2}[(1-w^*(\theta_1',\theta_2))t^*(\theta_1',\theta_2)+w^*(\theta_1',\theta_2)(p(\theta_1,\theta_2)-c_1)]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}_{G_2}[(1-w^*(\theta_1,\theta_2))t^*(\theta_1,\theta_2)+w^*(\theta_1,\theta_2)(p(\theta_1,\theta_2)-c_1)]\\
=&\mathbb{E}_{G_2}[(1-w^*(\theta_2|\theta_1))t^*(\theta_2|\theta_1)+w^*(\theta_2|\theta_1)(p(\theta_1,\theta_2)-c_1)]\\
=&U^*_{1,m^*(\theta_1),\theta_1}\\
\geq&\bar{p}_1(\theta_1)-c_1
\end{aligned}$$

and therefore (w^*, t^*) is incentive compatible and individually rational for 1. Together, this establishes that (w^*, t^*) is incentive feasible and safe. Optimality of problem (36) then implies that (w^*, t^*) is undominated among all incentive feasible protocols. Therefore, (w^*, t^*) is a strong solution. ■

As shown in Proposition 7 and Proposition 8. Under common value environment, the main feature in the private value environment still holds. That is, when additional private information is introduced to the party with full bargaining power, it is possible to attain the efficiency level as if such private information were publicly observable. Although Proposition 8 is slightly weaker than Proposition 4 in the private value environment in the sense that probability of breakdown in any strong solution when both parties have private information might not be the same as when the signal of the principal were observable in general, from Proposition 1, restricting attention on the most possibly efficient strong solution which yields the same ex-ante probability of breakdown as if the principal's signal were public is seems to be a reasonable selection criterion. Since the interim payoffs for the principal under any strong solution are the same, it is natural to impose the restriction to the principal for her to choose the most efficient strong solution if there are many.[25]

5 Extension: Crisis Bargaining with Intermediate Bargaining Powers

Throughout the discussions above, we have been examining the cases where one of the party has full bargaining power and design optimal bargaining protocols in terms of their own preferences. To establish a more complete statement about the relationship among information, bargaining power and efficiency, it is crucial to consider situations under which both parties have some degree of bargaining power and then bargaining over the underlying protocol accordingly. In this section, we further extend the framework developed above to incorporate *any* kinds of allocation of bargaining powers. The introduction of intermediate bargaining powers would then allow us to understand the effect of alignment between bargaining power and information structure more completely, instead of considering extreme allocation of bargaining powers only.

Since we have been aiming for establishing a framework to study crisis bargaining without specifying the underlying bargaining protocols, it is not obvious how should the bargaining over underlying protocols with intermediate bargaining powers be formulated. Without specifying and particular details of the bargaining protocol, following the spirit of the Nash bargaining solution (Nash, 1950), we can approach this problem through axiomatic descriptions on the *solution concepts*. Myerson (1984) provided an extension of Nash's bargaining solution under environments with incomplete information, which is referred as the *neutral bargaining solutions*. We will adopt this solution concept and apply this to the environment of crisis bargaining.[26] In brief, a neutral bargaining solution is the smallest correspondence that maps any well-defined Bayesian bargaining problems into sets of incentive feasible protocols that satisfy three axioms: *the probability invariant axiom*, *the extension axiom* and the *random dictatorship axiom*. To formally define this solution concept, we first need some more general notations and definitions. Let Θ_1, Θ_2 be (measurable)

[25] The reason why uniqueness cannot be generally ensured in common value environment is that for a general function $p : [0, 1] \to [0, 1]$, the marginal gains under incentive compatibility constraints also depend on the marginal contributions of signals to relative power. That is, the sub-gradient of equilibrium payoff is a product of marginal gain of relative power and probability of winning and the probability of breakdown for player 1. This then creates some potential problems in identifying probability of breakdowns.

[26] Essentially, Myerson (1984) only provided the solution concept and characterizations in cases with finite types. We will extend this concept to infinite types in our frameworks.

type spaces. For each $i \in \{1,2\}$, let F_i be a map from Θ_i to the set of CDFs on Θ_{-i}. That is, for each $\theta_i \in \Theta_i$, $F_i(\cdot|\theta_i)$ describes a conditional belief of individual i on the other's type. Furthermore, let $\mathbb{D} = (D, \mathcal{D})$ be a measurable space[27] which denotes the set of alternatives and let $d_0 \in D$ denote the alternative that indicates that bargaining breaks down.[28] Finally, for each $i \in \{1,2\}$, let $u_i : D \times \Theta_i \to \mathbb{R}$ be a measurable function that describes the ex-post payoffs for i. With these elements, a *Bayesian bargaining problem* Γ can be defined by:

$$\Gamma = (\mathbb{D}, d_0, \Theta_1, \Theta_2, F_1, F_2, u_1, u_2)$$

A bargaining protocol in a Bayesian bargaining problem Γ can then be defined as a map from (reported) types to the measures on the outcomes D. Formally,

$$m : \Theta \to \Delta(D, \mathcal{D})$$

is a bargaining protocol, where $m(\cdot|\theta) \in \Delta(D, \mathcal{D})$ can be interpreted as a distribution of outcomes given that the individuals report $\theta = (\theta_1, \theta_2)$. To connect this more general framework to the previous settings, notice that in the previous bargaining problems, $\mathbb{D} = ([0,1]^2, \mathcal{B})$, where $\mathcal{B}$ is the Borel algebra on $[0,1]^2$. A generic element in D is then a pair (w,t) with $d_0 = (1,t)$ for some $t \in [0,1]$. The protocols considered above, which are deterministic, are simply dirac measures on the set D for each given θ. As conventional, given a Bayesain bargaining problem $\Gamma = (\mathbb{D}, d_0, \Theta_1, \Theta_2, F_1, F_2, u_1, u_2)$, and a bargaining protocol m, we write the interim payoff under truth-telling strategies for player i as:

$$U_i^*(\theta_1|m) := \int_{\Theta_{-i}} \int_D u_i(\delta, \theta) m(d\delta|\theta) F_i(d\theta_{-i}|\theta_i).$$

As before, we say that a protocol m is incentive compatible if for any $i \in \{1,2\}$, $\theta_i, \theta_i' \in \Theta_i$,

$$U_i^*(\theta_i|m) \geq \int_{\Theta_{-i}} \int_D u_i(\delta, \theta) m(d\delta|\theta_i', \theta_{-i}) F_i(d\theta_{-i}|\theta_i).$$

Also, we say that a protocol m is individually rational if for any $i \in \{1,2\}$, $\theta_i \in \Theta_i$,

$$U_i^*(\theta_i|m) \geq \int_{\Theta_{-i}} u_i(d_0, \theta) F_i(d\theta_{-i}|\theta_i).$$

We say the a protocol is incentive feasible if it is incentive compatible and individually rational. Analogously, a protocol is safe for $-i$ if it is ex-post incentive compatible and individually rational

[27] With $\mathcal{D}$ being a sigma algebra on D

[28] For simplicity, we will assume henceforth that $\{d_0\} \in \mathcal{D}$

for i. That is, m is safe for $-i$ if for any $\theta_i, \theta_i' \in \Theta_i$ and any $\theta_{-i} \in \Theta_{-i}$,

$$\int_D u_i(\delta,\theta)m(d\delta|\theta) \geq \int_D u_i(\delta,\theta)m(d\delta|\theta_i',\theta_{-i}),$$
$$\int_D u_i(\delta,\theta)m(d\delta|\theta) \geq u_i(d_0,\theta).$$

We will also say that a protocol m is *interim efficient* if it is incentive feasible and there is no incentive feasible protocol m' such that $U_i^*(\theta_i|m') \geq U_i^*(\theta_i|m)$ for all $i \in \{1,2\}$, any $\theta_i \in \Theta_i$, with strict inequality for some $i \in \{1,2\}$, some $\theta_i \in \Theta_i$. Furthermore, we say that a Bayesian bargaining problem $\bar{\Gamma} = (\bar{\mathbb{D}}, d_0, \Theta_1, \Theta_2, F_1, F_2, \bar{u}_1, \bar{u}_2)$ is an *extension* of $\Gamma = (\mathbb{D}, d_0, \Theta_1, \Theta_2, F_1, F_2, u_1, u_2)$ if $\bar{\mathbb{D}} = (\bar{D}, \bar{\mathcal{D}})$, $D \subset \bar{D}$, $\mathcal{D} \subset \bar{\mathcal{D}}$ and $\bar{u_i}|_D \equiv u_i$ for each $i \in \{1,2\}$. That is, $\bar{\Gamma}$ is an extension of Γ if the set of outcomes in the extension contains the set of outcomes in the original problem and the payoff functions behave the same on the original set. With these general settings, we are then able to state the axioms and define the neutral bargaining solutions formally.

Let S be a correspondence that maps a Bayesian bargaining problem into the collection of sets of incentive feasible protocols in that problem, That is, $S(\Gamma)$ is a subset of incentive feasible protocols under Γ. We can interpret S as a *solution concept* for Bayesian bargaining problems. The following three axioms about S will be crucial for the definition of neutral bargaining solutions.

Definition 6. A solution concept S satisfies the *probability invariant axiom* if for any two Bayesian bargaining problems $\Gamma = (\mathbb{D}, d_0, \Theta_1, \Theta_2, F_1, F_2, u_1, u_2)$ and $\hat{\Gamma} = (\mathbb{D}, d_0, \Theta_1, \Theta_2, \hat{F}_1, \hat{F}_2, \hat{u}_1, \hat{u}_2)$ such that:

$$\int_A u_i(d,\theta)F_i(d\theta_{-i}|\theta_i) = \int_A \hat{u}_i(d,\theta)\hat{F}_i(d\theta_{-i}|\theta_i)$$

for any measurable $A \subset \Theta_{-i}$, $d \in D$ and $i \in \{1,2\}$,

$$S(\Gamma) = S(\hat{\Gamma}).$$

Since the Bayesian bargaining problems are dealing with the cases where θ_{-i} is unknown to i, all the relative payoffs are essentially in the interim sense. The probability invariant axiom is then requiring that if the baseline elements for computing expected payoffs are the same in any to Bayesian bargaining problems, the associated solutions given by this solution concept must be the same.

Definition 7. A solution concept S satisfies the *extension* axiom if for any sequence of Bayesian bargaining protocols $\{\Gamma^k\}$ that are extensions of a Bayesian bargaining problem Γ. If $m^k \in S(\Gamma^k)$

for all $k \in \mathbb{N}$ and the induced interim payoffs $U_i^k(\cdot|m^k)$ in each Bayesian bargaining problem Γ^k satisfies:

$$\limsup_{k\to\infty} U_i^k(\theta_i|m^k) \leq U_i^*(\theta_i|m)$$

for all $\theta_i \in \Theta_i$, and $i \in \{1,2\}$, for some interim efficient protocol m, then $m \in S(\Gamma)$.

The extension axiom requires that if a protocol can generate payoffs that is greater than infinitely many of payoffs given by the solutions in some extensions for each individual and for each type, then such protocol must also be a solution in the original problem.

Definition 8. A solution concept S satisfies the *random dictatorship* axiom with respect to $\phi \in [0,1]$ if for any strong solutions m_1 for 1 and m_2 for 2, let $m^\phi := \phi m_1 + (1-\phi)m_2$. If m^ϕ is interim efficient, then $m^\phi \in S(\Gamma)$.[29]

As remarked above, in the bargaining problems with incomplete information, if a strong solution exists for some individual, then such choice would be the most reasonable one since it perfectly balances the trade-offs between optimality condition on one's own private information and information revelation. The random dictatorship axiom can be interpreted as follows: Suppose that both parties have strong solutions. Now suppose that they are randomly assigned the right to design the underlying protocol according to probabilities ϕ and $1-\phi$, respectively. It is then natural to expect that whenever one is assigned to the right, she will select her strong solution. This axiom is then requiring that a protocol that is expected to be selected must be a solution given by the solution concept S whenever it is also interim efficient. Notice that since a protocol specifies a probability measure on D given a reported types θ, any convex combination of protocols must also be a probability measure given θ and hence m^ϕ must be well-defined. Notice that this axiom imposes restrictions on S only when strong solutions exist for both parties and the convex combination m^ϕ is interim efficient. When there does not exist any strong solution for some individual or the convex combination of them is not interim efficient, this axiom is not restrictive.

With these axioms, we can now formally define a neutral bargaining solution. Given $\phi \in [0,1]$, let $\mathcal{S}^\phi$ be the collection of all solution concepts that satisfy the probability invariant axiom, the extension axiom and the random dictatorship axiom respect to ϕ.

[29]In the original statement in Myerson (1984), random dictatorship axiom is only defined with respect to $\phi = \frac{1}{2}$. Since we are interested in different allocation of bargaining powers, we extend the definition to any arbitrary non-trivial probability of assigning dictators to capture the idea of bargaining powers.

Definition 9. Given a Bayesian bargaining problem Γ and $\phi \in [0,1]$, the set of *ϕ-neutral bargaining solution* is given by

$$\bar{S}^{\phi}(\Gamma) := \bigcap_{S \in \mathcal{S}^{\phi}} S(\Gamma).$$

That is, for any Bayesian bargaining problem Γ, the set of ϕ-neutral bargaining solution is the smallest set identified by all the solution concepts that satisfy the probability invariant axiom, the extension axiom, and the random dictatorship axiom with respect to ϕ. From the random dictatorship axiom, it is natural to interpret $(\phi, 1-\phi)$ as the *allocation of bargaining power* for the neutral bargaining solution correspondence $\bar{S}^{\phi}$, since ϕ and $1-\phi$ are exactly the probability that one can be assigned full rights to design the protocols.

Recall from the previous sections, in all the environments except for one-sided incomplete information in common value environment, strong solutions for both parties exist. Furthermore, by Lemma B1, B2 and B4 in the appendix, any convex combination of strong solutions are interim efficient since strong solutions are interim efficient. Under the random dictatorship axiom, it is clear that the set of neutral bargaining solutions is non-empty under these environments. In fact, the following proposition shows that the convex combinations of interim payoffs of these strong solutions are the only possible interim payoffs in neutral bargaining solutions. This characterization would then facilitate us to compare the efficiencies under intermediate allocation of bargaining powers as well.

Proposition 9. *Let Γ be a Bayesian bargaining problem and let m_1 and m_2 be strong solutions for 1 and 2 under Γ, respectively. Let $m^{\phi} := \phi m_1 + (1-\phi) m_2$. Suppose that m^{ϕ} is interim efficient. Then for any ϕ-neutral bargaining solution $m \in \bar{S}^{\phi}(\Gamma)$,*

$$U_i^*(\cdot|m^{\phi}) \equiv U_i(\cdot|m).$$

Recall that Proposition 1 gives equivalent relationships among strong solutions in terms of interim payoffs. In the same spirit, Proposition 9 shows that whenever strong solutions exist for both parties and the convex combination of them with weights ϕ and $1-\phi$ is interim efficient, all the ϕ-neutral bargaining solutions must yield the same interim payoffs for both parties as that convex combination gives. This equivalence relationship can then enable us to extend the previous descriptions of the effect of alignment between information structure and allocation of bargaining power on efficiency, from extreme allocation of bargaining powers to any intermediate allocations. In the private value environments, strong solutions always exist, provided that Assumptions 1, 2 or

3 hold under their relevant environments. Also, from Lemma B1 and Lemma B2 in the appendix, any convex combinations of strong solutions is interim efficient and therefore Proposition 9 applies. When incomplete information is only one-sided, say, only c_1 is unknown to 2. Under Assumptions 1 and 2, let $m_1^* = (w_1^*, t_1^*)$ and $m_2^* = (w_2^*, t_2^*)$ be strong solutions for 1 and 2 respectively. By Proposition 2 and Proposition 3, $w_1^* \equiv 0$ and $w_2^*(c_1) = 0$ if $c_1 + c_2 \geq \frac{1-F_1(c_1)}{f_1(c_1)}$, $w_2^*(c_1) = 1$ if $c_1 + c_2 < \frac{1-F_1(c_1)}{f_1(c_1)}$. Let $m^\phi := \phi m_1^* + (1-\phi) m_2^*$. Lemma B1 in the appendix and Proposition 9 then give that for any ϕ-neutral bargaining solution m, $U_i^*(\cdot|m) \equiv U_i^*(\cdot|m^\phi)$ for each $i \in \{1, 2\}$. Lemma 2 and Lemma 3 then give that in any ϕ-neutral bargaining solution, $m = (w, t)$,

$$\mathbb{E}[w(c_1)] = \phi\mathbb{E}[w_1^*(c_1)] + (1-\phi)\mathbb{E}[w_2^*(c_1)] = (1-\phi)\mathbb{E}[w_2^*(c_1)] = (1-\phi)\mathbb{P}\left(\left\{c_1 + c_2 < \frac{1-F_1(c_1)}{f_1(c_1)}\right\}\right).$$

Similarly, when there is two-sided incomplete information, under Assumption 3, the (essentially unique) strong solutions for 1 and 2, $m_1^* = (w_1^*, t_1^*), m_2^* = (w_2^*, t_2^*)$, according to Proposition 4, have

$$w_1^*(c_1, c_2) = \begin{cases} 1, & \text{if } c_1 + c_2 - \frac{1-F_2(c_2)}{f_2(c_2)} \\ 0, & \text{otherwise} \end{cases}$$

and

$$w_2^*(c_1, c_2) = \begin{cases} 1, & \text{if } c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} \\ 0, & \text{otherwise} \end{cases}$$

By Lemma B2 in the appendix, $\phi m_1 + (1-\phi) m_2$ is interim efficient. From the payoff equivalence properties given by Proposition 1 and Proposition 9, and the fact that $\bar{w_1}^*$ and $\bar{w_2}^*$ are derivatives of $U_1^*(\cdot|m_1^*)$ and $U_2^*(\cdot|m_2^*)$ respectively, in any ϕ-neutral bargaining solution $m = (w, t)$,

$$\begin{aligned} \mathbb{E}[w(c_1, c_2)] &= \phi\mathbb{E}[w_1^*(c_1, c_2)] + (1-\phi)\mathbb{E}[w_2^*(c_1, c_2)] \\ &= \phi\mathbb{P}\left(\left\{c_1 + c_2 < \frac{1-F_2(c_2)}{f_2(c_2)}\right\}\right) + (1-\phi)\mathbb{P}\left(\left\{c_1 + c_2 < \frac{1-F_1(c_1)}{f_1(c_1)}\right\}\right). \end{aligned}$$

By the same reasons, in the common value environment where there is two-sided incomplete information, under Assumption 5 and 6, Proposition 1, Proposition 8, Lemma B4 in the appendix and Proposition 9 give that for neutral bargaining solution $m = (w, t)$, let $m_1^* = (w_1^*, t_1^*)$, $m_2^* = (w_2^*, t_2^*)$ be strong solutions for 1 and 2 respectively. Then

$$\begin{aligned} &\mathbb{E}[w(\theta_1, \theta_2)] = \phi\mathbb{E}[w_1^*(\theta_1, \theta_2)] + (1-\phi)\mathbb{E}[w_2^*(\theta_1, \theta_2)] \\ &= \phi\mathbb{P}\left(\left\{c_1 + c_2 + p_{\theta_2}(\theta_1, \theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)} < 0\right\}\right) + (1-\phi)\mathbb{P}\left(\left\{c_1 + c_2 - p_{\theta_1}(\theta_1, \theta_2)\frac{G_1(\theta_1)}{g_1(\theta_1)} < 0\right\}\right). \end{aligned}$$

With these observations, we can then extend our previously established results to cases with intermediate bargaining powers as well. Consider first the case where there is only one-sided incomplete information in private value environments, we have seen in Proposition 2 that the outcome will be ex-post efficient if 1, who has full information, is assigned full bargaining power whereas the probability of breakdown is positive if 2 is assigned full bargaining power instead. For the cases intermediate allocation of bargaining power, given the bargaining power of 1, $\phi \in (0,1)$, the above observation shows that the probability of breakdown in any neutral bargaining solution is $(1-\phi)\mathbb{P}(\{c_1 + c_2 < \frac{1-F_1(c_1)}{f_1(c_1)}\})$, which is greater than 0—the probability of breakdown when 1 has full bargaining power and less than $\mathbb{P}(\{c_1 + c_2 < \frac{1-F_1(c_1)}{f_1(c_1)}\})$—the probability of breakdown when 2 has full bargaining power. Therefore, we have the same relationship among information, efficiency and bargaining power even if bargaining powers are intermediate. That is, whenever the allocation of bargaining power is more well-aligned with the underlying information structure (larger ϕ), the ex-ante probability of breakdowns and thus efficiency level will be higher, with extreme allocations being the best and the worst scenarios. On the other hand, if there is two-sided incomplete information, either in private value or common value environments. Although it is less obvious in these environments how to describe the underlying information structure due to the fact that both parties have some private information, the observation above still enables us to extend the results to intermediate bargaining power cases. In the private value case, if we rewrite

$$\mathbb{P}\left(\left\{c_1 + c_2 < \frac{1-F_2(c_2)}{f_2(c_2)}\right\}\right) = \mathbb{P}\left(\left\{-c_1 > c_2 - \frac{1-F_2(c_2)}{f_2(c_2)}\right\}\right),$$

and

$$\mathbb{P}\left(\left\{c_1 + c_2 < \frac{1-F_1(c_1)}{f_1(c_1)}\right\}\right) = \mathbb{P}\left(\left\{-c_2 > c_1 - \frac{1-F_1(c_1)}{f_1(c_1)}\right\}\right),$$

then we can see that the probability of breakdowns in strong solutions depend on one's relative magnitude of private information and the other's *virtual types*. If the one with full bargaining power has smaller costs of breakdowns relative to the other's virtual costs, which can be interpreted as the magnitude of costs together with the information rents, in the ex-ante sense, then the probability of breakdown would be larger. This captures two features, first, probabilities of breakdowns of one's strong solution depends on relative magnitude of costs, if one's cost is essentially smaller then the other's ex-ante, he would be more aggressive when designing protocols since breakdowns cost less to him. The second is the magnitude of information rents. If the opponent's information rent is higher, it then means that she has more valuable private information and thus the informational advantage of the one designing protocols would be less. This is congruent with the previous observation that probability of breakdown is higher when information structure and bargaining power are mis-aligned, since the less advantage in terms of information that one has, the more likely that breakdown will occur. Similarly, under common value environment, we can also rewrite

$$\mathbb{P}\left(\left\{c_1 + c_2 + p_{\theta_2}(\theta_1, \theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)} < 0\right\}\right) = \mathbb{P}\left(\left\{-c_1 > c_2 + p_{\theta_2}(\theta_1, \theta_2)\frac{G_2(\theta_2)}{g_2(\theta_2)}\right\}\right)$$

and

$$\mathbb{P}\left(\left\{c_1 + c_2 - p_{\theta_1}(\theta_1, \theta_2)\frac{G_1(\theta_1)}{g_1(\theta_1)} < 0\right\}\right) = \mathbb{P}\left(\left\{-c_1 > c_2 - p_{\theta_1}(\theta_1, \theta_2)\frac{G_1(\theta_1)}{g_1(\theta_1)}\right\}\right).$$

The relationships are similar to the common value case, except that now the advantage of information is measured by the distributions of the corresponding virtual values $-p_{\theta_2}(\theta_1, \theta_2)\frac{G_2(\theta_1)}{g_2(\theta_2)}$ and $p_{\theta_1}(\theta_1, \theta_2)\frac{G_1(\theta_1)}{g_1(\theta_1)}$. These measures, not only depend on the underlying distributions G_1 and G_2, but also depend on the distributions of the marginal effects on relative power, p_{θ_1}, p_{θ_2}. For higher opponent's virtual values, which could be due to stronger marginal effects or higher $\frac{G_1}{g_1}$ and $\frac{G_2}{g_2}$, the probability of breakdown would be higher in one's strong solution. This is also congruent with the conclusion that probability of breakdown is higher when the information structure and/or the relative powers are mis-aligned with the allocation of bargaining powers.

Together, in the previous sections, we have shown that the ex-ante probability of inefficient breakdown is smaller if the allocation of bargaining power is better aligned with the underlying information structure in the cases of extreme allocation of bargaining powers. With intermediate bargaining powers, this is still true. From the observation above, under the private value environment, for any $\phi \in (0, 1)$, any ϕ-neutral bargaining solution $m = (w, t)$,

$$\mathbb{E}[w(c_1, c_2)] = \phi\mathbb{E}[w_1^*(c_1, c_2)] + (1 - \phi)\mathbb{E}[w_2^*(c_1, c_2)],$$

where w_1^* and w_2^* are given by strong solutions for 1 and d 2 respectively, which is a convex combination of the probability of breakdown in the strong solutions when each party has full bargaining power. Notice that this then implies

$$\min\{\mathbb{E}[w_1^*(c_1, c_2)], \mathbb{E}[w_2^*(c_1, c_2)]\} < \mathbb{E}[w(c_1, c_2)] < \max\{\mathbb{E}[w_1^*(c_1, c_2)], \mathbb{E}[w_2^*(c_1, c_2)]\}.$$

Therefore, when bargaining power is fully assigned to the party with more advantageous information and/or costs, that is, the one whose strong solution gives lower probability of breakdown, the outcome would be the most efficient among all other possible allocation (including intermediate) of bargaining powers. Therefore, we can still conclude that probability of breakdown is higher when the information structure and allocation of bargaining power is less well-aligned, with consideration of intermediate bargaining powers. In fact, one of the extreme allocations will be the most efficient, depending on which side has more advantageous information and cost structures. Furthermore, the more bargaining power this party has, the more efficient the outcome will be. Similar results also hold in common value environments, since the probability of breakdowns is also a convex combination of probability of breakdowns given by strong solutions for each party with weights $(\phi, 1-\phi)$.

The previous discussions focused on the cases where strong solution exists. However, in common value environment with one-sided incomplete information, strong solution does not exist and therefore the random dictatorship axiom cannot be directly applied to ensure existence of neutral bargaining solutions. The next proposition justifies the analogy of using convex combinations of solutions in the cases of full bargaining powers when allocation of bargaining power is intermediate. Specifically, it shows that for any *undominated* protocols for 1 and the strong solution for 2, the convex combination of them with weight $(\phi, 1-\phi)$ is indeed a ϕ-neutral bargaining solution.

Proposition 10. *Suppose that $p \sim F_1$ with density f_1 and that only 1 can observe p. Then for any $\phi \in [0,1]$, for any undominated deterministic incentive feasible protocol m_1 for player 1 and the strong solution m_2 for player 2, $m^\phi := \phi m_1 + (1-\phi)m_2$ is a ϕ-neutral bargaining solution.*

However, contrary to the cases when strong solution exists, we do not have full characterization of the interim payoffs under neutral bargaining solutions. Nevertheless, since any neutral bargaining solution must be interim efficient, and since any protocol with $w(p) > 0$ for some $p < p^*$ is dominated, as shown in the proof of Proposition 5, at least we can be ensured that the ex-ante probability of breakdown is always less than that when 2 has full bargaining power, $1 - F_1(p^*)$. Together, from the perspective of Proposition 10, the results before still generalizes when intermediate allocation of bargaining power is introduced. When bargaining power and information structure is less well-aligned, i.e. when ϕ is smaller, (ex-ante) probability of breakdown is higher, regardless of whether $\mathbb{E}_{F_1}[p] + c_2 \geq 1 - c_1$ or not and this probability is minimized under the given environment when 1 has full bargaining power. On the other hand, for each fixed $\phi \in [0,1]$, the (ex-ante) probability of breakdown is lower when $\mathbb{E}_{F_1}[p] + c_2 \geq 1 - c_1$, which means that that distribution of power still matters, as long as $\phi > 0$. Furthermore, even if we are not restricting to the neutral bargaining solutions given by Proposition 10, as remarked above, for any $\phi \in [0,1)$, any ϕ-neutral bargaining solution must give a lower (ex-ante) probability of breakdown than when player 2 has full bargaining power. Contrarily, since in the strong solution for 2, (ex-post) probability of breakdown is 1 whenever $p > p^*$, for any ϕ-neutral bargaining solution, there must exist an undominated protocol for 1 under which the ex-ante probability of breakdown is smaller. Therefore, it is always more efficient to assign full bargaining power to the party with full information, as in the private value case.

To sum up, from the above extensions, we can see that the main results in the previous sections—A bargaining outcome will be more efficient when the information structure and the allocation of bargaining power is more well-aligned—is still valid when intermediate allocation of bargaining power is also considered. The main implication is that, since the outcome is more efficient when more bargaining power is assigned to the party with "better" information, the optimal allocation of bargaining power is always extreme. That is, assigning full bargaining power to the party with better information is always (one of) the optimal allocation of bargaining power. This implication relies on the fact that interim efficient frontier is convex in each environment, as shown in section B in the appendix. As the interim efficient frontier is convex, convex combination of optimal choices for two parties when they have full bargaining power is exactly (one of) the solution when allocation of bargaining power is intermediate, such linearity then gives the optimality of extreme allocation. Intuitively, since probability of breakdown in optimal choices depend only on the distribution of players' *virtual types*, as remarked above, which can also be interpreted as *information rent*, it is always optimal to assign full bargaining power to the party whose opponent has less information rent ex-ante.

6 Discussion

Combining the previous results, there are two major implications. First, under the crisis bargaining framework, it is not incomplete information *alone* that causes inefficient breakdowns. Instead, it is the *alignment* between the allocation of bargaining power and the information structure that affects the efficiency. Specifically, if the is only one-sided incomplete information and the private information is private value, then whenever the allocation of bargaining power is perfectly aligned with the information structure—if the party with full bargaining power also has full information, then ex-post efficiency outcome in which probability of breakdown is always zero can still be achieve despite the presence of incomplete information. On the other hand, if the allocation of bargaining power is not aligned with the information structure—if the party with full information does not have full bargaining power, then under fairly weak conditions, there will be positive probability of inefficient breakdowns. Furthermore, the probability of breakdowns increases as the bargaining power of the uninformed party increases. On the other hand, when there is two-sided incomplete information, under any allocation of bargaining power, under some weak conditions, probability of breakdown will always be positive. However, the smallest probability of breakdown among all allocations of bargaining powers can be achieved when the party with more advantageous information—the one who's opponent has less information rent—is assigned to full bargaining power and such probability increases if more bargaining power is assigned to the party with less advantageous information. Similar feature also occurs under common value environments, except that in such environments, the alignment between allocation of bargaining power and relative power also matters. Therefore, under common value environments with one-sided incomplete information, even when allocation of bargaining power and information structure is perfectly aligned, there will still be positive probability of inefficient breakdowns if the party with full bargaining power does not have enough relative power ex-ante. However, as we have shown above, the distortion caused by misalignment between bargaining power and relative power is always less perverse than the distortion caused by misalignment between bargaining power and information. In brief, in common values environments with one-sided incomplete information, if bargaining power is well-aligned with both relative power and information structure, then ex-post efficiency can still be achieved despite the presence of incomplete information. If it is misaligned with relative power but well-aligned with information, there will be some efficiency losses. If it is misaligned with information, there will be even more efficiency losses. Similarly, when there is two-sided incomplete information, probability of breakdown increases if the one with less advantageous position, which is determined by both relative power and information structure, is assigned to more bargaining powers and the smallest probability of breakdown among these allocations is achieved by assigning all bargaining power to the party with more advantageous position. The second implication, which is closely related to the first, is that when additional private information is introduced to the party with full bargaining power, there will not be any extra efficiency loss and the probability of breakdown will be the same as if such private information were publicly observable. This means that the common proposition in the literature that introduction of private information causes distortion in crisis bargaining is in fact only partially valid. The analyses above indicate that *only* when private information is introduced to the party whose opponent has some bargaining power would there be extra distortions and the distortion is more significant when the party to whom private information is introduced has less bargaining power.

As remarked at the beginning, the result above does not rely on any *a priori* specifications of the underlying bargaining protocols and thus are robust to the details of all model-dependent assumptions. As long as we interpret bargaining power as in Myerson (1983, 1984)—the party of full bargaining power has the right to select the protocols according to her preference as long as it is feasible in terms of the information and both parties' interest under such information and intermediate allocations of bargaining powers are defined by randomly assigning such right, we can understand the relationship between information and efficiency as well as how and whether incomplete information causes distortions in a systematic way. On one hand, this approach maintains the desired generality of the second category in the literature in a "game-free" way. On the other hand, it further provides sharper descriptions about the role of incomplete information in affecting and determining efficiency in crisis bargaining, which was not yet achieved by the second category. For instance, in both private value and common value environments, the necessary and sufficient conditions for *existence* of always peaceful protocols provided in Fey & Ramsey (2011) are in fact the conditions for the *emergence* of such protocols whenever bargaining power and information are well-aligned. On the other hand, when the bargaining power and information structure are not perfectly aligned or incomplete information is two-sided so that a party with some level of bargaining powers faces some incomplete information, the *risk-return-trade-off* will prevail as in Fearon's (1995) simple take-it-or-leave-it specification and hence causes positive probability of breakdown. However, the probability of breakdown is less when the party with more advantageous position has more bargaining power. In sum, by taking a stand on the interpretation of bargaining power which does not depend on game forms, the above analyses provided an intermediate approach between the two main categories in the literature and enables us to obtain sharper and more specific descriptions as well as explanations about the effect of incomplete information on efficiency in a systematic, game-free way.

There are some practical implications regarding to the study of international conflicts can be drawn from these results. First, it is common—particularly in international crises—that , unlike the standard mechanism design environments, there is no credible or capable third party that can serve as an unbiased "designer" whose objective is to attain overall efficiency. In other words, as remarked in Horner et al. (2015), in international crises bargaining, it is unlikely that there exists an unbiased *arbitrator* who is capable of credibly enforce and bargaining protocols to maintain peace. As such, standard mechanism design approach is limited in analyzing the question of how to achieve efficiency in crisis bargaining. Various possible ways are proposed in the literature to address this problem. For instance, since it is commonly claimed that incomplete information causes distortion, one commonly suggested way is to reduce the amount of uncertainties, possibly through summits, arms control, or Confidence Building Measures (CBM). However, these measures are often costly and hard to be implemented. More importantly, by the same reason why incomplete information causes distortions, it is often against one's incentives to reveal private information truthfully. Another possible way is to introduce an *mediator*. Horner et al. (2015) showed that there are ways for the mediator to achieve efficiency level as if she were an arbitrator. However, this is conditioning on the premise that this mediator can commit both not to reveal any information during the mediation and to possibly mix the suggestions, which is also sometimes hard to be achieved by a mediator. The results above, however, suggests yet another possible way. As a "planner", which might be some international organization or some third party whose objective is to minimize the probability of war in a certain crisis bargaining and can credibly assign bargaining powers (via endorsement, for example) but cannot credibly design and enforce protocols on their own,[30] even if the capability and credibility of designing and enforcing a protocol is limited, one can still achieve the lowest possible probability of war by assigning the right of proposing bargaining protocols "correctly"—to the party who is the most informed. By correctly assigning bargaining powers, based on the previous results, one can still achieve the highest possible efficiency even if one cannot reduce the amount of uncertainties nor credibly commit to be an unbiased mediator.

The second application of the results above is to provide a more formal foundation of the *power transition theory*. In the power transition theory, it is claimed that the probability of war between two countries increases as their power reaches to parity (Organski, 1958). The reason is that when there is remarkable discrepancy between two countries' power, the weaker would never attempt to challenge the stronger. Contrarily, when the powers reach to parity, the *unsatisfied* country would have large incentive to initiate the war in order to change the status-quo (Organski & Kugler, 1980). As we can see in this argument, there are in fact two decisive factors on the probability of war—the level of satisfaction and parity of power. To certain degrees, the results above in fact provide a formal foundation of such reasoning. Specifically, if we interpret the level of satisfaction as how well the allocation of bargaining power reflects the distribution of power and information and decompose the transition to power parity into two components: change of distribution of power and change in information, in the sense that when powers reach to parity, one of the party's

[30]For instance, one can think of the UN security council trying to resolve some territorial disputes or a country trying to mediate a civil war of its neighbor.

relative power deteriorates and becomes more uncertain at the same time, then we can see from the previous results that such transitions indeed increase the probability of war. Starting from the scenario where the party with considerably high relative power and full information has full bargaining power. As powers reach to parity, distribution of power and information becomes less in favor of this party. Since allocation of bargaining power is unchanged throughout the transition phase, probability of war increases as a result.

Although throughout the discussions above, we have been focusing on international crisis bargaining, the analyses and results in fact are not restricted to such context. As long as the bargaining context exhibits the two features as crisis bargaining—no pre-determined bargaining protocols and voluntary agreements, the analyses above can be applied. For instance, we can also use this framework to study the bargaining between a rebel group and an incumbent government when the rebellion is significant to a degree that the original constitution is not effectively enforced. Likewise, we can also study legislative or constitutional bargaining in a newly formed democracy, where the part of the objects being bargained are protocols themselves by default.

7 Conclusion

In this paper, we examined the relationship between information and efficiency in the crisis bargaining framework. An intermediate approach is developed and is used to provider further descriptions and explanations about whether and how incomplete information causes distortions in crisis bargaining is a systematic way. In brief, the results suggest that incomplete information *alone* does not necessarily cause efficiency loss, neither is it the only factor that determines the level of distortion. Instead, it is the *mis-alignment* between allocation of bargaining power and information structure, and (possibly, when uncertainties are common value) the *mis-alignment* between the allocation of bargaining power and the distribution of power that causes inefficiencies and determines the level of distortions. Moreover, they also suggest that introduction of private information causes distortion only when it is introduced to the party without full bargaining power. However, there are several drawbacks of this approach and there are also some aspects that can be extended as well. First, the applicability of this intermediate approach is limited to contexts in which there are no obvious bargaining protocols at the first place. Litigation bargaining outside of the court, for instance, may not be suitable for this approach as there are in fact some pre-determined and effectively enforced protocols. In addition, it is also desirable to move beyond the standard quasi-linear and risk neutral preferences as well as totally ordered, one-dimensional type spaces to bargaining with possibly more general preferences and multidimensional types. Finally, we believe that it is of both theoretical and practical importance to develop a theory of *endogenous mechanism design*, where the designers are themselves a subset of the players of the designed mechanism and also have private information. This paper is in fact a special case, in which a particular payoff structure and number of players are specified. It would be of our future interest to extend this framework to a more general setting that is not restricted to crisis bargaining-alike contexts with more general preference and information structures.

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Appendix

A. Omitted Proofs

This section of the appendix provides the proofs that are omitted in the main text. We will omit the proofs of Lemma 1, Lemma 2 and Lemma 6, since they are special cases of Lemma 3 and Lemma 7.

Proof of Lemma 3. For necessity, suppose that $m = (w, t)$ is an incentive feasible protocol. Individual rationality gives that

$$U_1^*(c_1|m) \geq p - c_1, \forall c_1 \in [\underline{c_1}, \overline{c_1}] \tag{A1}$$

$$U_2^*(c_2|m) \geq 1 - p - c_2, \forall c_2 \in [\underline{c_2}, \overline{c_2}]. \tag{A2}$$

Incentive compatibility implies that for each $c_1 \in [\underline{c_1}, \overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$

$$U_1^*(c_1|m) = \operatorname*{argmax}_{c_1' \in [\underline{c_1}, \overline{c_1}]} U_1(c_1', c_1|m)$$

$$U_2^*(c_2|m) = \operatorname*{argmax}_{c_2' \in [\underline{c_2}, \overline{c_2}]} U_2(c_1', c_1|m)$$

where

$$U_1(c_1', c_1|m) = \mathbb{E}_{F_2}[(1 - w(c_1', c_2))t(c_1', c_2) + w(c_1', c_2)(p - c_1)]$$

$$U_2(c_1', c_1|m) = \mathbb{E}_{F_1}[(1 - w(c_1, c_2'))(1 - t(c_1, c_2')) + w(c_1, c_2')(1 - p - c_2)]$$

and hence for each $i \in \{1, 2\}$, $c_i' \in [\underline{c_i}, \overline{c_i}]$, $U_i^*(c_i', \cdot|m)$ is differentiable almost everywhere and maximizers exist since it is continuous on compact set $[\underline{c_i}, \overline{c_i}]$. Moreover,

$$\left|\frac{\partial}{\partial c_i} U_i^*(c_i', c_i|m)\right| = |\bar{w}_i(c_i')| \leq 1$$

for each $i \in \{1, 2\}$, $c_i, c_i' \in [\underline{c_i}, \overline{c_i}]$. Therefore, by the envelope theorem (see Milgrom & Segal (2002), Theorem 2),

$$U_1^*(c_1|m) = U_1^*(\underline{c_1}|m) - \int_{\underline{c_1}}^{c_1} \bar{w}_1(x)dx, \forall c_1 \in [\underline{c_1}, \overline{c_1}] \tag{A3}$$

$$U_2^*(c_2|m) = U_2^*(\underline{c_2}|m) - \int_{\underline{c_2}}^{c_2} \bar{w}_2(x)dx \forall c_2 \in [\underline{c_2}, \overline{c_2}] \tag{A4}$$

and

$$\bar{w}_1, \bar{w}_2 \quad \text{are non-increasing} \tag{A5}$$

since $U_1^*(\cdot|m)$ and $U_2^*(\cdot|m)$ are convex. Together, (A1)-(A5) gives conditions 1 to 3. As for condition 4, use (A3) and (A4) to rewrite (11) and then use the Fubini theorem and change of measure to write

$$\begin{aligned}
&\mathbb{E}_{F_i}\left[\int_{\underline{c_i}}^{c_i} \bar{w}_i(x)dx\right] \\
&= \int_{\underline{c_{-i}}}^{\overline{c_{-i}}} \int_{\underline{c_i}}^{\overline{c_i}} \int_{\underline{c_i}}^{c_i} w_i(x, c_{-i}) f_i(c_i) dx dc_i dF_{-i}(c_{-i}) \\
&= \int_{\underline{c_{-i}}}^{\overline{c_{-i}}} \int_{\underline{c_i}}^{\overline{c_i}} \int_{x}^{\overline{c_i}} f_i(c_i) dc_i w_i(x, c_{-i}) dx dF_{-i}(c_{-i}) \\
&= \int_{\underline{c_{-i}}}^{\overline{c_{-i}}} \int_{\underline{c_i}}^{\overline{c_i}} w_i(x, c_{-i})(1 - F_i(x)) dx dF_{-i}(c_{-i}) \\
&= \int_{\underline{c_{-i}}}^{\overline{c_{-i}}} \int_{\underline{c_i}}^{\overline{c_i}} w_i(c_i, c_{-i}) \left(\frac{1 - F_i(c_i)}{f_i(c_i)}\right) dF_i(c_i) dF_{-i}(c_{-i}),
\end{aligned}$$

for each $i \in \{1, 2\}$, where $w_1(c_1, c_2) := w(c_1, c_2)$ and $w_2(c_2, c_1) := w(c_1, c_2)$ for any $c_1 \in [\underline{c_1}, \overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$. Condition 4 then follows. As for conditions 5 and 6, notice that (A3) and (A4) gives

$$\begin{aligned}
U_1^*(\underline{c_1}|m) &= \mathbb{E}_{F_2}[(1 - w(c_1, c_2))t(c_1, c_2) + w(c_1, c_2)(p - c_1)] + \int_{\underline{c_1}}^{c_1} \bar{w}_1(x)dx \\
U_2^*(\underline{c_2}|m) &= \mathbb{E}_{F_1}[(1 - w(c_1, c_2))(1 - t(c_1, c_2)) + w(c_1, c_2)(1 - p - c_2)] + \int_{\underline{c_2}}^{c_2} \bar{w}_2(x)dx,
\end{aligned}$$

for any $c_1, \in [\underline{c_1}, \overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$. Using the fact that $0 \leq t \leq 1$, we then have

$$\bar{w}_1(c_1)(p - c_1) + \int_{\underline{c_1}}^{c_1} \bar{w}_1(x)dx \leq U_1^*(\underline{c_1}|m) \leq 1 - \bar{w}_1(c_1) + \bar{w}_1(c_1)(p - c_1) + \int_{\underline{c_1}}^{c_1} \bar{w}_1(x)dx \tag{A6}$$

$$\bar{w}_2(c_2)(1 - p - c_2) + \int_{\underline{c_2}}^{c_2} \bar{w}_2(x)dx \leq U_1^*(\underline{c_2}|m) \leq 1 - \bar{w}_2(c_2) + \bar{w}_2(c_2)(1 - p - c_2) + \int_{\underline{c_2}}^{c_2} \bar{w}_2(x)dx, \tag{A7}$$

for all $c_1, \in [\underline{c_1}, \overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$ Notice that since $\bar{w}_1$ and $\bar{w}_2$ are non-increasing $p \in [0, 1]$, and $\underline{c_1}, \underline{c_2} > 0$, the lower bounds in (A6) and (A7) are non-increasing and the upper bounds in (A6) and (A7) are non-decreasing and hence (A6) and (A7) holds for all $c_1, \in [\underline{c_1}, \overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$ if and only if conditions 5 and 6 holds.

For sufficiency, suppose that $(U_1^*(\underline{c_1}), U_2^*(\underline{c_2}), w)$ satisfies conditions 1 to 6. From the equivalence between (A6) and (A7) for all $c_1, \in [\underline{c_1}, \overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$ and conditions 5 and 6, conditions 4-6 then ensures that there exists some $t : [\underline{c_1}, \overline{c_1}] \times [\underline{c_2}, \overline{c_2}] \to [0, 1]$ such that

$$U_1^*(\underline{c_1}) - \int_{\underline{c_1}}^{c_1} \bar{w}_1(x)dx = \mathbb{E}_{F_2}[(1 - w(c_1, c_2))t(c_1, c_2) + w(c_1, c_2)(p - c_1)] \tag{A8}$$

$$U_2^*(\underline{c_2}) - \int_{\underline{c_2}}^{c_2} \bar{w}_2(x)dx = \mathbb{E}_{F_1}[(1 - w(c_1, c_2))(1 - t(c_1, c_2)) + w(c_1, c_2)(1 - p - c_2)], \tag{A9}$$

for any $c_1 \in [\underline{c_1}, \overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$. Let $m := (w, t)$. Then by (A8),(A9) and conditions 2 and 3, for any $c_1', c_1 \in [\underline{c_1}, \overline{c_1}]$,[31]

$$\begin{aligned}
&U_1(c_1', c_1|m) = \mathbb{E}_{F_2}[(1 - w(c_1', c_2))t(c_1', c_2) + w(c_1', c_2)(p - c_1)] \\
=&U_1^*(c_1'|m) + \bar{w}_1(c_1')(c_1' - c_1) \\
=&U_1^*(c_1|m) - \int_{c_1}^{c_1'} (\bar{w}_1(x) - \bar{w}_1(c_1'))dx \\
\leq&U_1^*(c_1|m),
\end{aligned}$$

where the last inequality follows from $\bar{w}_1$ being non-increasing. Therefore m is incentive compatible for 1. Similar arguments hold for 2. Finally, (A8), (A9) and conditions 2 and 3 also imply that m is individually rational. Together, $m = (w, t)$ is incentive feasible. This completes the proof. ■

Proof of Lemma 4. For necessity, let $m = (w, t)$ be any incentive feasible protocol. Individual rationality gives that

$$U_1^*(p|m) \geq p - c_1, \forall p \in [0, 1] \tag{A10}$$

$$U_{2,m}^* \geq 1 - \mathbb{E}_{F_1}[p] - c_2, . \tag{A11}$$

Incentive compatibility implies that for each $p \in [0, 1]$

$$U_1^*(p|m) = \underset{p' \in [0,1]}{\operatorname{argmax}} U_1(p', p|m),$$

where

$$U_1(p', p|m) = (1 - w(p'))t(p') + w(p')(p - c_1)$$

and hence for each $p \in [0, 1]$, $U_1(p', \cdot|m)$ is differentiable almost everywhere and maximizers exist since it is continuous on compact set $[0, 1]$. Moreover,

$$\left| \frac{\partial}{\partial p} U_i(p', p|m) \right| = |w(p')| \leq 1$$

for each $p \in [0, 1]$. Therefore, by the envelope theorem,

$$U_1^*(p|m) = U_1^*(1|m) - \int_p^1 w(x)dx, \forall p \in [0, 1] \tag{A12}$$

and

$$w \quad \text{is non-increasing} \tag{A13}$$

[31] Here we follow the Riemann integral convention that the integral operators are denoted such that $\int_{c_1}^{c_1'} = -\int_{c_1}^{c_1'}$

since $U_1^*(\cdot|m)$ is convex. Together, (A10)-(A13) gives conditions 1 to 3. As for condition 4, use (A12) to rewrite the recourse constraint and then use the Fubini theorem and change of measure to write

$$\mathbb{E}_{F_1}\left[\int_p^1 w(x)dx\right] = \int_0^1 \int_p^1 w(x)f_1(p)dxdp$$
$$= \int_0^1 \int_0^x f_1(p)dpw(x)dx = \int_0^1 w(x)F_1(x)dx$$
$$= \int_0^1 w(p)\frac{F_1(p)}{f_1(p)}dF_1(p).$$

Condition 4 then follows. As for conditions 5, notice that (A12)

$$U_1^*(1|m) = (1 - w(p))t(p) + w(p)(p - c_1) + \int_p^1 w(x)dx$$

for any $p, \in [0, 1]$. Using the fact that $0 \leq t \leq 1$, we then have

$$w(p)(p - c_1) + \int_p^1 w(x)dx \leq U_1^*(1|m) \leq 1 - w(p) + w(p)(p - c_1) + \int_p^1 w(x)dx \tag{A14}$$

for all $p \in [0, 1]$. Notice that since w is non-decreasing and $c_1, c_2 > 0$, the lower bound in (A14) is non-decreasing and the upper bound in (A14) is non-increasing and hence (A14) holds for all $p \in [0, 1]$ if and only if condition 5 holds.

For sufficiency, suppose that $(U_1^*(1), U_2^*, w)$ satisfies conditions 1 to 5. From the equivalence between (A14) for all $p \in [0, 1]$ and condition 5 , conditions 4 and 5 then ensures that there exists some $t : [0, 1] \to [0, 1]$ such that

$$U_1^*(1) - \int_p^1 w(x)dx = (1 - w(p))t(p) + w(p)(p - c_1) \tag{A15}$$

$$U_2^* = \mathbb{E}_{F_1}[(1 - w(p))(1 - t(p)) + w(p)(1 - p - c_2)], \tag{A16}$$

for any $p \in [0, 1]$. Let $m := (w, t)$. Then by (A15),(A16) and condition 2, for any $p', p \in [0, 1]$,

$$U_1(p', p|m) = (1 - w(p'))t(p') + w(p')(p - c_1)$$
$$=U_1^*(p'|m) - w(p')(p' - p)$$
$$=U_1^*(p|m) + \int_p^{p'} (w(x) - w(p'))dx$$
$$\leq U_1^*(p|m),$$

where the last inequality follows from w being non-decreasing. Therefore m is incentive compatible for 1. Finally, (A13) and conditions 2 and 3 also imply that m is individually rational. Together, $m = (w, t)$ is incentive feasible. This completes the proof. ■

Proof of Lemma 5. From conditions 2, 3 and 4 of Lemma 4, for any $m \in \mathcal{M}_4$,

$$U_1^*(1|m) \leq \mathbb{E}_{F_1}[p] + c_2 - \int_0^1 w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF(p).$$

Condition 5 of Lemma 4 also gives

$$U_1^*(1|m) \leq (1 - w(1)) + w(1)(1 - c_1).$$

Let $\mathcal{W}$ denote the set of increasing functions that map from $[0,1]$ to $[0,1]$. Then

$$V(1) = \max_{w \in \mathcal{W}} \min\left\{\mathbb{E}_{F_1}[p] + c_2 - \int_0^1 w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF(p), (1 - w(1)) + w(1)(1 - c_1)\right\}. \tag{A18}$$

Notice first that the functional

$$w \mapsto \min\left\{\mathbb{E}_{F_1}[p] + c_2 - \int_0^1 w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF(p), (1 - w(1)) + w(1)(1 - c_1)\right\}$$

is convex. Also $\mathcal{W}$ is compact under the topology induced by $L^1([0,1])$ norm by Helley's selection theorem and the Lebesgue dominant convergence theorem and is convex as well. Therefore, there exists some extreme points of $\mathcal{W}$ that attain the maximum in (A18). Since the set of extreme points of $\mathcal{W}$ is the set of functions of form:

$$w(p) = \begin{cases} 1, & \text{if } p > p_0 \\ 0, & \text{otherwise} \end{cases} \tag{A19}$$

for some $p_0 \in [0,1]$, (A18) can be rewritten into

$$V(1) = \max_{p^* \in [0,1]} \min\left\{\mathbb{E}_{F_1}[p] + c_2 - \int_0^1 w^*(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF(p), (1 - w^*(1)) + w^*(1)(1 - c_1)\right\},$$

where

$$w^*(p) := \begin{cases} 1, & \text{if } p > p^* \\ 0, & \text{otherwise} \end{cases},$$

for any $p^* \in [0,1]$. Notice that

$$\begin{aligned}
&\max_{p^* \in [0,1)} \min\left\{\mathbb{E}_{F_1}[p] + c_2 - \int_0^1 w^*(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF(p), (1 - w^*(1)) + w^*(1)(1 - c_1)\right\} \\
=& \max_{p^* \in [0,1)} \min\left\{\mathbb{E}_{F_1}[p] + c_2 - \int_0^1 w^*(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF(p), (1 - c_1)\right\} \\
=& 1 - c_1.
\end{aligned}$$

Also, if $p^* = 1$,

$$\begin{aligned}
&\min\left\{\mathbb{E}_{F_1}[p] + c_2 - \int_0^1 w^*(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF(p), (1 - w^*(1)) + w^*(1)(1 - c_1)\right\} \\
&= \min\{\mathbb{E}_{F_1}[p] + c_2, 1\} \\
&= \mathbb{E}_{F_1}[p] + c_2.
\end{aligned}$$

Therefore, if $\mathbb{E}_{F_1}[p] + c_2 \geq 1 - c_1$, from (A19) and the above observation, $V(1) = \mathbb{E}_{F_1}[p] + c_2$. Similarly, if $\mathbb{E}_{F_1}[p] + c_2 < 1 - c_1$, $V(1) = 1 - c_1$. Together,

$$V(1) = \max\{\mathbb{E}_{F_1}[p] + c_2, 1 - c_1\}.$$

■

Proof of Lemma 7. For necessity, let $m = (w, t)$ be an incentive feasible protocol. Individually rationality gives that

$$U_1^*(\theta_1|m) \geq \bar{p_1}(\theta_1) - c_1, \forall \theta_1 \in [0, 1] \tag{A20}$$

$$U_2^*(\theta_2|m) \geq 1 - \bar{p_2}(\theta_2) - c_2, \forall \theta_2 \in [0, 1]. \tag{A21}$$

Incentive compatibility implies that for each $\theta_1, \theta_2, \in [0, 1]$,

$$\begin{aligned}
U_1^*(\theta_1|m) &= \operatorname*{argmax}_{\theta_1' \in [0,1]} U_1(\theta_1', \theta_1|m) \\
U_2^*(\theta_2|m) &= \operatorname*{argmax}_{\theta_2' \in [0,1]} U_2(\theta_2', \theta_2|m),
\end{aligned}$$

where

$$\begin{aligned}
U_1(\theta_1', \theta_1|m) &= \mathbb{E}_{G_2}[(1 - w(\theta_1', \theta_2))t(\theta_1', \theta_2) + w(\theta_1', \theta_2)(p(\theta_1, \theta_2) - c_1)] \\
U_2(\theta_2', \theta_2|m) &= \mathbb{E}_{G_1}[(1 - w(\theta_1, \theta_2'))(1 - t(\theta_1, \theta_2')) + w(\theta_1, \theta_2')(1 - p(\theta_1, \theta_2) - c_2)]
\end{aligned}$$

for any $\theta_1', \theta_1, \theta_2', \theta_2 \in [0, 1]$ and hence for each $i \in \{1, 2\}$, $U_i(\theta_1', \cdot|m)$ is differentiable since p_{θ_1}, p_{θ_2} exist and are continuous. (so that differentiation and expectation can be interchanged) Moreover,

$$\left|\frac{\partial}{\partial \theta_i} U_i(\theta_i', \theta_i)\right| = |\mathbb{E}_{G_{-i}}[w(\theta_1, \theta_2)p_{\theta_i}(\theta_1, \theta_2)]| \leq \sup_{\theta_{-i} \in [0,1]} |p_{\theta_i}(\theta_i|\theta_{-i})|,$$

for each $i \in \{1, 2\}$, $\theta_i, \theta_i' \in [0, 1]$, where $p_{\theta_1}(\theta_1|\theta_2) := p_{\theta_1}(\theta_1, \theta_2)$, $p_{\theta_2}(\theta_2|\theta_1) := -p_{\theta_2}(\theta_1, \theta_2)$, for all $\theta_1, \theta_2 \in [0, 1]$. By condition 4 in Assumption 5, $\sup_{\theta_{-i} \in [0,1]} |p_{\theta_i}(\cdot|\theta_{-i})|$ is integrable and thus by

Milgrom & Segal (2002), Theorem 2,

$$U_1^*(\theta_1|m) = U_1^*(1|m) - \int_{\theta_1}^{1} \bar{v_1}(x)dx, \forall \theta_1 \in [0,1] \tag{A22}$$

$$U_2^*(\theta_2|m) = U_2^*(1|m) - \int_{\theta_2}^{1} \bar{v_2}(x)dx, \forall \theta_2 \in [0,1] \tag{A23}$$

and

$$\bar{v_1}, \bar{v_2} \quad \text{are non-decreasing} \tag{A24}$$

since $U_1^*(\cdot|m)$, $U_2^*(\cdot|m)$ are convex. Together, (A20)-(A24) gives conditions 1 to 3. As for condition 4, use (A22) and (A23) to rewrite the recourse constraint and then use the Fubini theorem and change of measure to write

$$\begin{aligned}
&\mathbb{E}_{G_i}\left[\int_{\theta_i}^{1} \bar{v_i}(x)dx\right] \\
&= \int_0^1 \int_0^1 \int_{\theta_1}^1 p_{\theta_i}(x|\theta_{-i})w_i(x,\theta_{-1})dxdG_i(\theta_i)dG_{-i}(\theta_{-i}) \\
&= \int_0^1 \int_0^1 \int_0^x g_i(\theta_i)d\theta_i p_{\theta_i}(x|\theta_{-i})w_i(x,\theta_{-1})dxdG_{-i}(\theta_{-i}) \\
&= \int_0^1 \int_0^1 p_{\theta_i}(x|\theta_{-i})w_i(x,\theta_{-1})G_i(\theta_i)d\theta_i dG_{-i}(\theta_1,\theta_2). \\
&= \int_0^1 \int_0^1 w_i(\theta_i,\theta_{-i})p_{\theta_i}(\theta_i|\theta_{-i})\frac{G_i(\theta_i)}{g_i(\theta_i)}dG_i(\theta_i)dG_{-i}(\theta_{-i}),
\end{aligned}$$

for each $i \in \{1,2\}$. This gives condition 4. Conditions 5 and 6 follows from (A22) and (A23) and the equivalence relationship as in the proofs of Lemma 3 and Lemma 4.

For sufficiency, suppose that $(U_1^*(1), U_2^*(1), w)$ satisfies conditions 2 through 6 and that w is non-decreasing in both arguments. From the equivalence relationship as above, conditions 4 to 6 imply that there exists $t : [0,1]^2 \to [0,1]$ such that

$$U_1^*(1) - \int_{\theta_1}^1 \bar{v_1}(x)dx = \mathbb{E}_{G_2}[(1-w(\theta_1,\theta_2))t(\theta_1,\theta_2) + w(\theta_1,\theta_2)(p(\theta_1,\theta_2)-c_1)] \tag{A25}$$

$$U_2^*(1) - \int_{\theta_2}^1 \bar{v_2}(x)dx = \mathbb{E}_{G_1}[(1-w(\theta_1,\theta_2))(1-t(\theta_1,\theta_2)) + w(\theta_1,\theta_2)(1-p(\theta_1,\theta_2)-c_2)] \tag{A26}$$

for any $\theta_1, \theta_2 \in [0,1]$. Let $m := (w,t)$. Then by (A25), (A26), conditions 2 and 3, for any

$\theta_1', \theta_1 \in [0,1]$,

$$
\begin{aligned}
&U_1(\theta_1', \theta_1|m) = \mathbb{E}_{G_2}[(1-w(\theta_1,\theta_2)t(\theta_1,\theta_2) + w(\theta_1,\theta_2)(p(\theta_1,\theta_2)-c_1)] \\
=&U_1^*(\theta_1'|m) - \mathbb{E}_{G_2}[w(\theta_1',\theta_2)(p(\theta_1',\theta_2)-p(\theta_1,\theta_2))] \\
=&U_1^*(\theta_1|m) + \int_{\theta_1}^{\theta_1'} \bar{v}_1(x)dx - \mathbb{E}_{G_2}[w(\theta_1',\theta_2)(p(\theta_1',\theta_2)-p(\theta_1,\theta_2))] \\
=&U_1^*(\theta_1|m) + \int_{\theta_1}^{\theta_1'} (\bar{v}_1(x) - \mathbb{E}_{G_2}[p_{\theta_1}(x,\theta_2)w(\theta_1',\theta_2)])dx \\
=&U_1^*(\theta_1|m) + \int_{\theta_1}^{\theta_1'} \mathbb{E}_{G_2}[p_{\theta_1}(x,\theta_2)(w(x,\theta_2)-w(\theta_1',\theta_2))]dx \\
\leq&U_1^*(\theta_1|m),
\end{aligned}
$$

where the second equality follows from (A25), the third equality is from the Fundamental Theorem of Calculus, the forth equality follows from the definition of $\bar{v}_1$ and the inequality follows from monotonicity of w and that p is non-decreasing in its first argument. Also, (A25), (A26) and conditions 2 and 3 also imply that m is individually rational for 1. Similar arguments show that $U_2(\theta_2', \theta_2|m) \leq U_2^*(\theta_2|m)$ for any $\theta_2', \theta_2 \in [0,1]$ and m is individually rational for 2. Together, $m = (w,t)$ is incentive feasible, as desired. ■

Proof of Claim 1. For any $m, m' \in \mathcal{M}_4$, we say that m is *strictly dominated* by m' if $U_1^*(p|m) < U_1^*(p|m')$ for all $p \in [0,1)$. Then for any $m \in \mathcal{M}_4$, if m is strictly dominated by some $n \in \mathcal{M}_4$. By Lemma 5, we may without loss of generality suppose that $U_1^*(1|n) = 1-c_1$. Then there exists $\epsilon > 0$ such that any $m' \in \mathcal{M}_4$ with $\sup_{p\in[0,1]} |U_1^*(p|m') - U_1^*(p|m)| < \epsilon$ is also strictly dominated by n. By convexity of $U_1^*(\cdot|m)$ and the Markov inequality, there exists $\delta > 0$ such that for any $m' \in \mathcal{M}_4$ such that

$$
\int_0^1 |U_1^*(p|m') - U_1^*(p|m)|^2 dp < \delta^2,
$$

$U_1^*(p|m') < U_1^*(p|n)$ for all $p \in [0,1]$. Therefore, there exists $\delta > 0$ such that any $m' \in N_\delta(m) \cap \mathcal{M}_4$ is blocked by $n \in \mathcal{M}_4$. On the other hand, if $m \in \mathcal{M}_4$ is not strictly dominated, the proof of Proposition 5 shows that it must be that $w \equiv 0$ on $[0, p^*)$. Take $n \in \mathcal{M}_4$ that blocks m. As shown in the proof of Proposition 5, it must be that $U_1^*(p|m) \geq \beta(p)$ for all $p \in [0, \hat{p})$, since otherwise m would strictly dominated by $m_{\tilde{p}}$ for some $\tilde{p} \in [0, \hat{p})$. On the other hand, if $U_1^*(p|m) < U_1^*(p|\hat{m})$ for any $p \in [p', 1]$ for some $p' \in [\hat{p}, 1)$. From (24) and monotonicity of $U_1^*(\cdot|\hat{m})$, there exists $\epsilon > 0$ such that for any $m' \in \mathcal{M}_4$ such that $\sup_{p\in[0,1]} |U_1^*(p|m') - U_1^*(p|m)| < \epsilon$, m' is blocked by $\hat{m}$. Again by convexity of $U_1^*(\cdot|m)$ and the Markov inequality, there exists $\delta > 0$ such that for any

$m' \in N_\delta(m) \cap \mathcal{M}_4$, m' is blocked by $\hat{m}$. Now it suffices to check that for all $m \in \widehat{\mathcal{M}_4}$, the assertion holds. First observe the followings. For any non-decreasing $w : [0,1] \to [0,1]$, since $\frac{F_1}{f_1}$ is increasing, for any $p'' > p' > p^*$,

$$\begin{aligned}
&\mathbb{E}_{F_1}\left[w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \geq p''\right] \leq \mathbb{E}_{F_1}[p|p \geq p''] \\
\Rightarrow &\mathbb{E}_{F_1}\left[w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \geq p'\right] < \mathbb{E}_{F_1}[p|p \geq p'].
\end{aligned} \tag{A27}$$

Also, for any such function w, define

$$\underline{w}(p) := \begin{cases} w(p), & \text{if } p \in [p^*, 1] \\ 0, & \text{if } p \in [0, p^*) \end{cases}$$

Then again by monotonicity, for any $p' > p^*$,

$$\mathbb{E}_{F_1}\left[w(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \leq p'\right] > \mathbb{E}_{F_1}\left[\underline{w}(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \leq p'\right]. \tag{A28}$$

Furthermore, by construction, for any $p_0 \in [p^*, \hat{p}]$,

$$U^*_{2,m_{p_0}} = c_1 - (1 - F_1(p_0))\alpha(p_0)\mathbb{E}_{F_1}\left[c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\middle| p \geq p_0\right] = 1 - \mathbb{E}_{F_1}[p] - c_2.$$

Since $\frac{F_1(p)}{f_1(p)} > c_1 + c_2$ for all $p \in (p^*, 1]$, simple algebras imply that

$$c_1 - \alpha(p_0)\mathbb{E}_{F_1}\left[c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\middle| p \geq p_0\right] > 1 - \mathbb{E}_{F_1}[p|p \geq p_0] - c_2.$$

Together, again with some simple algebras, this implies that:

$$\beta(p_0)F(p_0) = \mathbb{E}_{F_1}[U^*_1(p|m_{p_0})|p \leq p_0]F(p_0) > \mathbb{E}_{F_1}[p] + c_2. \tag{A29}$$

Now consider any $m \in \widehat{\mathcal{M}_4}$ and suppose that there is some $n \in \mathcal{M}_4$ that blocks m. Let w, v be the corresponding sub-gradients of $U^*_1(\cdot|m)$ and $U^*_1(\cdot|n)$ respectively. We first claim that it cannot be the case that $U^*_1(p|n) > U^*_1(p|m)$ if and only if $p \in [0, p_0)$,

$$U^*_1(1|n) + \mathbb{E}_{F_1}\left[v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \leq p_0\right] = \mathbb{E}_{F_1}[p|p \leq p_0] + c_2$$

and

$$U^*_1(1|n) + \mathbb{E}_{F_1}\left[v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right] = \mathbb{E}_{F_1}[p] + c_2.$$

for some $p_0 \in (p^*, 1]$. Indeed, if so, then

$$U^*_1(1|n) + \mathbb{E}_{F_1}\left[v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \geq p_0\right] = \mathbb{E}_{F_1}[p|p \geq p_0] + c_2.$$

By (A27), for any $p' > p_0$,

$$U_1^*(1|n) + \mathbb{E}_{F_1}\left[v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \geq p'\right] < \mathbb{E}_{F_1}[p|p \geq p'] + c_2$$

and hence there exists $p_1 \in (p_0, 1]$ such that

$$U_1^*(1|n) + \mathbb{E}_{F_1}\left[v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \leq p_1\right] < \mathbb{E}_{F_1}[p|p \leq p_1] + c_2.$$

Since $[0, p_0] \subset [0, p_1]$, n cannot block m. Therefore, if $U_1^*(p|n) > U_1^*(p|m)$ if and only if $p \in [0, p_0)$ and n blocks m, it must be either

$$U_1^*(1|n) + \mathbb{E}_{F_1}\left[v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \leq p_0\right] < \mathbb{E}_{F_1}[p|p \leq p_0] + c_2,$$

or

$$U_1^*(1|n) + \mathbb{E}_{F_1}\left[v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\middle| p \leq p_0\right] = \mathbb{E}_{F_1}[p|p \leq p_0] + c_2,$$

and

$$U_1^*(1|n) + \mathbb{E}_{F_1}\left[v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right)\right] < \mathbb{E}_{F_1}[p] + c_2.$$

In either case,[32] there exists $\epsilon > 0$ and $\tilde{n} \in \mathcal{M}_4$ such that for any $m' \in \mathcal{M}_4$ with $\sup_{p \in [0,1]} |U_1^*(p|m') - U_1^*(p|m)| < \epsilon$, $\tilde{n}$ blocks m. Similarly, by convexity of $U_1^*(\cdot|m)$ and the Markov inequality, there exists $\delta > 0$ such that for any $m' \in N_\delta(m) \cap \mathcal{M}_4$, $\tilde{n}$ blocks m. Now consider the case where there is some $0 < p_0 < p_1 \leq 1$ such that $U_1^*(p|n) > U_1^*(p|m)$ for all $p \in [p_0, p_1]$. If there exists some $p' \in (0, p_0)$ such that $U_1^*(p|m) < U_1^*(p|n)$ if and only if $p \in [0, p'] \cup [p_0, p_1]$, since $U_1^*(\cdot|m)$ is constant on $[0, p^*]$, $p_0 > p' \geq p^*$. Also, by possibly replacing $U_1^*(p|n)$ with $U_1^*(p^*|n)$ for all $p \in [0, p^*)$,[33]. Since $p^* \in (0, \hat{p})$ and $U_1^*(p|m) \geq \beta(p)$ for all $p \in [0, \hat{p})$. (A29) implies that

$$1 - \mathbb{E}_{F_1}[U_1^*(p|n)|p \leq \tilde{p}] - \mathbb{E}_{F_1}[v(p)(c_1 + c_2)|p \leq \tilde{p}] > \mathbb{E}_{F_1}[p|p \leq \tilde{p}] + c_2,$$

for any $\tilde{p} \in [0, p^*]$, and hence

$$1 - \mathbb{E}_{F_1}[U_1^*(p|n)|p \geq \tilde{p}] - \mathbb{E}_{F_1}[v(p)(c_1 + c_2)|p \geq \tilde{p}] < \mathbb{E}_{F_1}[p|p \geq \tilde{p}] + c_2,$$

[32] By possibly "rotating" $U_1^*(\cdot|n)$ counter clock-wisely around p_0 if it is the second case. For instance, define $\tilde{v}(p)$ by $\alpha v(p)$ for all $p \in (p^*, 1]$ and $\tilde{v}(p) = v(p)$ otherwise for some $\alpha > 1$ such that $\alpha v(1) < 1$. This can be done since by the hypothesis here, $v < 1$ on $[0, 1]$. Then redefine $\tilde{n}$ by $U_1^*(p|\tilde{n}) := U_1^*(p_0|n) + \int_{p_0}^{p} \tilde{v}(x)dx$ for all $p \in [p_0, 1]$ and $U_1^*(p|\tilde{n}) - \int_{p}^{p_0} \tilde{v}(x)dx$ for all $p \in [0, p_0)$, with a choice of such α such that $U_1^*(1|\tilde{n}) < U_1^*(1|m)$. Then $\tilde{n}$ also blocks m and $\tilde{n}$ is in the first case as well.

[33] This is without loss of generality since $\mathbb{E}_{F_1}[c_1 + c_2 - \frac{F_1(p)}{f_1(p)}]$ is reduced by such change and the set where $U_1^*(p|n) > U_1^*(p|m)$ is unchanged since $p_0 > p' > p^*$

for any $\tilde{p} \in [0, p^*]$. Continuity and (A27) then implies that for any $\tilde{p} \in [0, 1]$,

$$1 - \mathbb{E}_{F_1}[U_1^*(p|n)|p \geq \tilde{p}] - \mathbb{E}_{F_1}[v(p)(c_1 + c_2)|p \geq \tilde{p}] < \mathbb{E}_{F_1}[p|p \geq \tilde{p}] + c_2. \tag{A30}$$

Then n is $[\tilde{p}, 1]$-individually rational for all $\tilde{p} \in [0, 1)$. Let ψ and ϕ be sub-gradients of $U_1^*(\cdot|n)$ at p' and p_0 respectively. Let

$$v_0(p) := \begin{cases} v(p), & \text{if } p \in (p_0, 1] \\ \phi, & \text{otherwise,} \end{cases}.$$

$$v_1(p) := \begin{cases} v(p), & \text{if } p \in [0, p') \\ \psi, & \text{otherwise} \end{cases}.$$

Take $\delta > 0$ such that $U_1^*(p'|m) - \delta < U_1^*(0|m)$ and define

$$g(p) := U_1^*(1|n) - \int_p^1 v_0(x)dx,$$
$$h(p) := U_1^*(0|n) - \delta + \int_0^p v_1(x)dx,$$

for all $p \in [0, 1]$, Let $U_1^*(p|\tilde{n}) := \max\{g(p), h(p)\}$ for all $p \in [0, 1]$. By construction, $U_1^*(p|\tilde{n})$ is increasing and convex, $U_1^*(p|\tilde{n}) > U_1^*(p|m)$ if and only if $p \in [p_0, p_1]$ and $\tilde{n}$ is Ω-individually rational for 2 for any $[p_0, p_1] \subseteq \Omega \subseteq [0, 1]$ by (A30) and by $p_0 > p' \geq p^*$. Therefore, we may without loss of generality suppose that $U_1^*(p|n) > U_1^*(p|m)$ if and only if $[p_0, p_1]$ since otherwise n would be in the case above as $U_1^*(\cdot|m)$ is constant on $[0, p^*]$. (A30) then implies that there exists some $\epsilon > 0$ such that for any $m' \in \mathcal{M}_4$ such that $\sup_{p \in [0,1]} |U_1^*(p|m') - U_1^*(p|m)| < \epsilon$, n blocks m'. Similarly, by convexity of m and the Markov inequality, there exists $\delta > 0$ such that for any $m' \in N_\delta(m) \cap \mathcal{M}_4$, n blocks m. Finally, if $p_0 \leq p^*$, then since $U_1^*(\cdot|m)$ is constant on $[0, p^*]$, it must be that $p_1 > p^*$ and $U_1^*(p|n) \leq U_1^*(p|m)$ for any $p \in [p_0, p^*]$. By replacing $U_1^*(p|n)$ with $U_1^*(p^*|n)$ for all $p \in [0, p^*)$, we then have the previous case with $p' = p^*$ and thus there exists $\delta > 0$ such that for all $m' \in N_\delta(m) \cap \mathcal{M}_4$, n blocks m. This then completes the proof. ■

Proof of Claim 2. For any $n, m \in \mathcal{M}_4$, define

$$\Omega(n, m) := \{p \in [0, 1] | U_1^*(p|n) > U_1^*(p|m)\}.$$

We will prove the claim by induction. Given any $\{n_k\}_{k=1}^K \subset \mathcal{M}_4$. Let $n^{(1)} := n_1$. Clearly, $n^{(1)} \in \mathcal{M}_4$ and $n^{(1)}$ is not blocked by n_1. Now suppose that there exists $k \in \{1, \ldots, K-1\}$, $n^{(k)} \in \mathcal{M}_4$ and $\mathcal{K}(k) \subset \{1, \ldots, K\}$ with $|\mathcal{K}(k)| = k$, such that $n^{(k)}$ is not blocked by any $n_j \in \{n_i\}_{i \in \mathcal{K}(k)}$. If $n^{(k)}$

is not blocked by any $n_j \in \{n_i\}_{i\in\{1,\ldots,K\}\setminus\mathcal{K}(k)}$, define $n^{(k+1)} := n^{(k)}$. Then $n^{(k+1)}$ is not blocked by any $n_j \in \{n_i\}_{i=1}^K$. In particular, there exists some $\mathcal{K}(k+1) \subset \{1,\ldots,K\}$ with $|\mathcal{K}(k+1)| = k+1$ and $n^{(k+1)}$ is not blocked by any $n_j \in \{n_i\}_{i\in\mathcal{K}(k+1)}$. If, on the other hand, $n^{(k)}$ is blocked by some $n_{k'} \in \{n_i\}_{i\in\{1,\ldots,K\}\setminus\mathcal{K}(k)}$, there are two cases. If $\Omega(n^{(k)}, n_{k'}) \neq \emptyset$ and $n^{(k)}$ is $\Omega(n^{(k)}, n_{k'})$-individually rational for 2, let $n^{(k+1)}$ be defined by

$$U_1^*(p|n^{(k+1)}) := \max\{U_1^*(p|n^{(k)}), U_1^*(p|n_{k'})\}.$$

Since $U_1^*(\cdot|n^{(k+1)})$ is a pointwise maximum of two convex functions, it is also convex. Together with the facts that $n_{k'}$ blocks $n^{(k)}$ and that $n^{(k)}$ is $\Omega(n^{(k)}, n_{k'})$-individually rational for 2, $n^{(k+1)} \in \mathcal{M}_4$ and is not blocked by any $n_j \in \{n_i\}_{i\in\mathcal{K}(k)\cup\{k'\}}$ by construction and by induction hypothesis. Let $\mathcal{K}(k+1) := \mathcal{K}(k) \cup \{k'\}$. If $\Omega(n^{(k)}, n_{k'}) = \emptyset$ or $n^{(k)}$ is not $\Omega(n^{(k)}, n_{k'})$-individually rational for 2. For any $j \in \mathcal{K}(k)$, if n_j blocks $n_{k'}$, it must be that $\Omega(n_{k'}, n^{(k)}) \subset \Omega(n_j, n_{k'})$, since otherwise n_j would block $n^{(k)}$, contradicting to the induction hypothesis. Therefore, $n_{k'}$ is $\Omega(n_j, n_{k'})$-individually rational for 2. Let $n^{(k+1)}$ be defined by:

$$U_1^*(p|n^{(k+1)}) := \max\{U_1^*(p|n_{k'}), U_1^*(p|n_i)\}_{i\in\widehat{\mathcal{K}(k)}},$$

where $\widehat{\mathcal{K}(k)} := \{i \in \mathcal{K}(k) | n_i \text{ blocks } n_{k'}\}$. Since $U_1^*(p|n^{(k+1)})$ is pointwise maximum of a finite collection of convex functions, it is also convex. Also, by construction, $n^{(k+1)}$ is individually for 2 and is not blocked by any $n_j \in \{n_i\}_{i\in\mathcal{K}(k)\cup\{k'\}}$. Let $\mathcal{K}(k+1) := \mathcal{K}(k) \cup \{k'\}$. Together, by induction, for each $k \in \{1,\ldots,K\}$, there exists $n^{(k)} \in \mathcal{M}_4$ and $\mathcal{K}(k)$ such that $|\mathcal{K}(k)| = k$ and that $n^{(k)}$ is not blocked by any $n_j \in \{n_i\}_{i\in\mathcal{K}(k)}$. Since $|\mathcal{K}(K)| = K$ and $\mathcal{K}(K) \subset \{1,\ldots,K\}$, $\mathcal{K}(K) = \{1,\ldots,K\}$. Thus, $n^{(K)} \in \mathcal{M}_4$ and is not blocked by any $n_j \in \{n_i\}_{i=1}^K$, as desired. ■

Proof of Proposition 9. First observe that, for any extension $\bar{\Gamma}$ as an extension of Γ. Suppose that for $i \in \{1,2\}$, $\bar{m}_i$ is a strong solution in $\bar{\Gamma}$ and that m_i is not a strong solution in $\bar{\Gamma}$. Notice that since $\bar{\Gamma}$ is an extension of Γ and $m_i(D|\theta) = 1$ for any $\theta \in \Theta$, $U_i^*(\cdot|m) \equiv \bar{U}_i(\cdot|m)$. By Proposition 1, it must be that $U_i^*(\cdot|m_i) \neq \bar{U}_i(\cdot|\bar{m}_i)$. Since $\bar{m}_i$ is a strong solution in $\bar{\Gamma}$, $\bar{U}_i(\cdot|\bar{m}_i)$ must not be dominated by $\bar{U}_i(\cdot|m_i)$. Now suppose that $\bar{U}_i(\cdot|m_i)$ is also not dominated by $\bar{U}_i(\cdot|\bar{m}_i)$. Let $\Omega := \{\theta_i \in \Theta_i | \bar{U}_i(\theta_i|m_i) > \bar{U}_i(\theta_i|\bar{m}_i)\}$. Then $\Omega \neq \emptyset$. Now define a protocol in $\bar{\Gamma}$, $\tilde{m}_i$ as follows:

$$\tilde{m}_i(A|\theta) := \begin{cases} m_i(A \cap D|\theta) & \text{if } \theta_i \in \Omega \\ \bar{m}_i(A|\theta) & \text{if } \theta_i \notin \Omega \end{cases}, \forall A \in \mathcal{D}.$$

Since m_i is safe for i in Γ and $\bar{m}_i$ is safe for i in $\bar{\Gamma}$, $\bar{\Gamma}$ is an extension of Γ, and $m(D|\theta) = 1$ for any $\theta \in \Theta$, $\tilde{m}_i$ is also safe in $\bar{\Gamma}$ by construction. Moreover, since $\Omega \neq \emptyset$, $\bar{U}_i(\theta_i|\tilde{m}_i) \geq U_i^*(\theta_i|\bar{m}_i)$ for all

$\theta_i \in \Theta_i$ and the inequality holds for some $\theta_i \in \Theta_i$. Thus, $\tilde{m}_i$ is safe and dominates $\bar{m}_i$, contradicting to $\bar{m}_i$ being a strong solution in $\bar{\Gamma}$. Therefore, if $\bar{m}_i$ is a strong solution for i in $\bar{\Gamma}$ and m_i is not a strong solution for i in $\bar{\Gamma}$, $\bar{m}_i$ must dominates m_i.

Now given a Bayesian bargaining problem with strong solutions m_1, m_2 for 1 and 2 respectively and suppose that $\phi m_1 + (1-\phi)m_2$ is interim efficient. Define a solution concept S as follows: For any Bayesian bargaining problem Γ', if Γ' is an extension of Γ or $\Gamma' = \Gamma$ and if the sets of strong solutions for both players in Γ' are non-empty, define $S(\Gamma')$ to be all the interim efficient protocols that yield the same interim payoffs as those generated by convex combinations of strong solutions for 1 and 2 in Γ' with weights $(\phi, 1-\phi)$, if any. Otherwise, let $S(\Gamma')$ be the set of interim efficient protocols. If Γ' is an extension of Γ but strong solution does not exist for either 1 or 2, define $S(\Gamma')$ to be all the incentive feasible protocols that yield the same interim payoffs as those generated by convex combinations of strong solutions for 1 and 2 with weights $(\phi, 1-\phi)$. Otherwise, define $S(\Gamma')$ to be all the interim efficient protocols. It is clear that S satisfies the probability invariant axiom since interim payoffs are unchanged. By construction, it is also clear that S satisfies the random dictatorship axiom with respect to ϕ. To verify that S satisfies the extension axiom, it suffices to show that whenever $U_i^*(\theta_i|m) \neq U_i^*(\theta_i|m^\phi)$ for some $i \in \{1,2\}$, $\theta_i \in \Theta_i$, there dose not exist any sequence of extensions, $\{\Gamma^k\}$ and protocols $\{m^k\}$, such that m^k is a convex combination of strong solutions for 1 and 2 with weights $(\phi, 1-\phi)$ in Γ^k for each $k \in \mathbb{N}$ and $\limsup_{k\to\infty} U_i^*(\theta_i|m^k) \leq U_i^*(\theta_i|m)$ for each $i \in \{1,2\}$, $\theta_i \in \Theta_i$. Indeed, for any sequence of extensions of Γ, $\{\Gamma^k\}$, and any sequences of convex combinations of strong solutions $\{m^k\}$ which are interim efficient, for each k, if m_1 and m_2 are also strong solutions in Γ^k for 1 and 2 respectively, Proposition 1 and the fact that $m_1(D|\theta) = m_2(D|\theta) = 1$ for all $\theta \in \Theta$ implies that $U_i^*(\cdot|m) \equiv U_i^k(\cdot|m^k)$. On the other hand, if either m_1 or m_2 are not strong solutions in Γ^k, the above observation and the fact that strong solutions are safe imply that for each $i \in \{1,2\}$,

$$U_i^*(\theta_i|m^\phi) \leq U_i^k(\theta_i|m^k),$$

for every $\theta_i \in \Theta_i$, with strict inequality for some $\theta_i \in \Theta_i$. Therefore, for any such sequence,

$$\limsup_{k\to\infty} U_i^k(\theta_i|m^k) \geq U_i^*(\theta_i|m^\phi),$$

for any $i \in \{1,2\}$, $\theta_i \in \Theta_i$. If furthermore, there exists some interim efficient protocol m in Γ such that

$$\limsup_{k\to\infty} U_i^k(\theta_i|m^k) \leq U_i^*(\theta_i|m),$$

for each $i \in \{1,2\}$ and $\theta_i \in \Theta_i$, then

$$U_i^*(\theta_i|m^\phi) \leq U_i(\theta_i|m),$$

for each $i \in \{1,2\}$ and $\theta_i \in \Theta_i$. This then implies that $U_i^*(\cdot|m) \equiv U_i^*(\cdot|m^\phi)$, whenever m^ϕ is interim efficient. Together, S satisfies the extension axiom as well. Since S satisfies all three axioms, any protocol m such that $U_i^*(\theta_i|m) \neq U_i^*(\theta_i|m^\phi)$ for some $i \in \{1,2\}$, $\theta_i \in \Theta_i$, is not in $S(\Gamma)$ and thus is not in $\bar{S}^\phi(\Gamma)$. ■

Proof of Proposition 10. Let m_1 be any undominated, deterministic and incentive feasible protocol for 1 and m_2 be the (essentially unique) strong solution for 2. The proof of Proposition 5 implies that $m_1 = \bar{m}$ if $\mathbb{E}_{F_1}[p] + c_2 \geq 1 - c_1$ and $m_1 \in \widehat{\mathcal{M}_4}$ if $\mathbb{E}_{F_1}[p] + c_2 < 1 - c_1$. On the other hand, Proposition 6 gives that

$$w_2(p) = \begin{cases} 1, & \text{if } c_1 + c_1 - \frac{F_1(p)}{f_1(p)} < 0 \\ 0, & \text{otherwise} \end{cases}.$$

Let Γ be the original Bayesian bargaining problem given by the environment. Consider a sequence of extensions of Γ, denoted by $\{\Gamma^k\}$, defined as follows. For each $k \in \mathbb{N}$, $D^k := [0,1]^2 \cup \{e\}$, endowed with the Borel algebra on $[0,1]^2 \cup \{e\}$, for some $e \in \mathbb{R}^2$. For any $p \in [0,1]$, let

$$u_1^k(d,p) := \begin{cases} (1-w)t + w(p-c_1), & \text{if } d = (w,t) \in [0,1]^2 \\ p - c_1 + \frac{1}{k} + \delta, & \text{if } d = e \end{cases}$$

for some $\delta \geq 0$ and

$$u_2^k(d,p) := \begin{cases} (1-w)(1-t) + w(1-p-c_2), & \text{if } d = (w,t) \in [0,1]^2 \\ 1 - p - c_2, & \text{if } d = e \end{cases}.$$

We first notice that under the extended Bayesian bargaining problems, the incentive feasible protocols can still be characterized in a similar way as in Lemma 4. Indeed, notice that for any $k \in \mathbb{N}$, for any protocol $\tilde{m} : [0,1] \to \Delta(D^k)$, payoff for 1 when having signal p and reporting p' is

$$U_1^k(p',p|\tilde{m}) := \mu(p') \int_{[0,1]^2} ((1-w)t + w(p-c_1)\tilde{m}_D(dw,dt|p') + (1-\mu(p'))(p - c_1 + \frac{1}{k} + \delta),$$

where $\mu(p) := \tilde{m}([0,1]^2|p)$ and $\tilde{m}_D(\cdot|p)$ is the conditional measure given by $\tilde{m}(\cdot|p)$ on $D = [0,1]^2$. Since $U_1^k(p',\cdot|\tilde{m})$ is differentiable almost everywhere and $|\frac{\partial}{\partial p} U_1^k(p',p|\tilde{m})| = |\mu(p')\bar{w}(p') + (1 - \mu(p'))| \leq 1$ for all $p \in [0,1]$, the envelope theorem applies and hence

$$U_1^k(p|\tilde{m}) = U_1^k(1|\tilde{m}) - \int_p^1 v(x)dx, \forall p \in [0,1], \tag{A31}$$

where $v(p) := \mu(p)\bar{w}(p) + (1-\mu(p))$ and is non-decreasing, $\bar{w}(p) := \int_{[0,1]^2} w\tilde{m}_D(dw,dt|p)$, for all $p \in [0,1]$. On the other hand,

$$\begin{aligned}
&\mathbb{E}_{F_1}[U_1^k(p|\tilde{m})] + U_{2,\tilde{m}}^k\\
=&\int_0^1 \big(\mu(p)(1-\bar{w}(p)(c_1+c_2)) + (1-\mu(p))(p-c_1+\frac{1}{k}+\delta+1-p-c_2)\big)dF_1(p)\\
=&1-\int_0^1 v(p)(c_1+c_2)dF_1(p) + (1-\bar{\mu})\left(\frac{1}{k}+\delta\right), \qquad \text{(A32)}
\end{aligned}$$

where $\bar{\mu} := \int_0^1 \mu(p)dF_1(p)$. Together, by interchanging integral and Fubini's theorem, (A31) and (A32) yields

$$U_1^k(1|\tilde{m}) + U_{2,\tilde{m}}^k = 1 - \int_0^1 v(p)\left(c_1+c_2-\frac{F_1(p)}{f_1(p)}\right)dF_1(p) + (1-\bar{\mu})\left(\frac{1}{k}+\delta\right). \qquad \text{(A33)}$$

The rest of the proof is organized as follows: Consider any solution concept S and any $\phi \in [0,1]$ that satisfies that three axioms with allocation of bargaining power $(\phi, 1-\phi)$. For each $k \in \mathbb{N}$, we will first find strong solutions m_1^k and m_2^k for 1 and 2, respectively. Lemma B3 then ensures that for any $\phi \in [0,1]$, the convex combination of these strong solutions with weight $(\phi, 1-\phi)$ is interim efficient. The random dictatorship axiom then requires that $\phi m_1^k + (1-\phi)m_2^k \in S(\Gamma^k)$. On the other hand, we will show that $\lim_{k\to\infty} U_1^k(p|m_1^k) \le U_1^*(p|m_1)$, $\lim_{k\to\infty} U_1^k(p|m_2^k) = U_1^k(p|m_2)$, $U_{2,m_1^k}^k = U_{2,m_1}^*$ for all $k \in \mathbb{N}$ and $\limsup_{k\to\infty} U_{2,m_2^k}^k \le U_{2,m_2}^*$. It then follows that the limits of payoffs yielded by these convex combinations are below the payoffs yielded by $\phi m_1 + (1-\phi)m_2$. Therefore, by the extension axiom, $\phi m_1 + (1-\phi)m_2 \in S(\Gamma)$. Since such S is arbitrary, it then follows that $\phi m_1 + (1-\phi)m_2 \in \bar{S}^\phi$, as desired.

Consider first the case where $1-c_1 > \mathbb{E}_{F_1}[p]+c_2$. In this case, set $\delta = 0$. We will first show that for each $k \in \mathbb{N}$, m_1^k defined as $m_1^k(\{e\}|p) = 1$ for all $p \in [0,1]$, is a strong solution for 1. Clearly, since $u_2^k(e,p) = 1-p-c_2$, m_1^k is safe. It then suffices to show that m_1^k is undominated. Notice that for any $k \in \mathbb{N}$, Lemma 5 ensures that there is no incentive feasible protocol $m \in \mathcal{M}_4$ that can dominates $p - c_1 + \frac{1}{k}$ in the original Bayesian bargaining problem. Suppose that m_1^k is dominated by some $\tilde{m} : [0,1] \to \Delta(D^k)$ in Γ^k that is incentive feasible. By the characterization above,

$$U_1^k(p|\tilde{m}) = U_1^k(1|\tilde{m}) - \int_p^1 v(x)dx$$

for some non-decreasing $v : [0,1] \to [0,1]$. Since v is non-decreasing, by Lemma 4, we can define an incentive feasible protocol $\hat{m} \in \mathcal{M}_4$ by $(U_1^*(1|\hat{m}), U^2_{2,\hat{m}}, v)$, where

$$U^*_{2,\hat{m}} = 1 - \mathbb{E}_{F_1}[p] - c_2$$
$$U_1^*(1|\hat{m}) = 1 - U^*_{2,\hat{m}} - \int_0^1 v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF_1(p).$$

Since $U_1^k(p|m_1^k) = p - c_1 + \frac{1}{k}$ is undominated for each $k \in \mathbb{N}$ in the original Bayesian bargaining problem Γ, for any $k \in \mathbb{N}$, there exists $p^k \in [0,1]$ such that

$$U_1^k(1|\hat{m}) - \int_{p^k}^1 v(x)dx < p^k - c_1 + \frac{1}{k}. \tag{A34}$$

On the other hand, since $\tilde{m}$ dominates m_1^k in the Bayesain bargaining problem Γ^k, and $\tilde{m}$ is incentive feasible, it is without loss of generality to suppose that $U^k_{2,\tilde{m}} = 1 - \mathbb{E}_{F_1}[p] - c_2$. [34] Dominance implies that for all $p \in [0,1]$,

$$U_1^k(1|\tilde{m}) - \int_p^1 v(x) \geq p - c_1 + \frac{1}{k}.$$

Using $U^k_{2,\tilde{m}} = 1 - \mathbb{E}_{F_1}[p] - c_2$ and (A33), this can be rewritten as:

$$\begin{aligned}
&\mathbb{E}_{F_1}[p] + c_2 - \int_0^1 v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF_1(p) + (1-\bar{\mu})\frac{1}{k} - \int_p^1 v(x)dx \geq p - c_1 + \frac{1}{k} \\
\Leftrightarrow &\mathbb{E}_{F_1}[p] + c_2 - \int_0^1 v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF_1(p) - \int_p^1 v(x)dx \geq p - c_1 + \bar{\mu}\frac{1}{k},
\end{aligned}$$

for all $p \in [0,1]$. If $\bar{\mu} = 0$, it must be that $\tilde{m}(e|p) = 1$ for F_1-almost all p and hence $\tilde{m}$ cannot dominate m_1^k since F_1 is absolute continuous. If $\bar{\mu} \in (0,1]$, then there exists $k' \in \mathbb{N}$ such that $\frac{1}{k'} < \bar{\mu}\frac{1}{k}$. Take such k', and use the definition of $U_1^*(1|\hat{m})$, we then have:

$$U_1^*(1|\hat{m}) - \int_p^1 v(x)dx \geq p - c_1 + \frac{1}{k'},$$

for all $p \in [0,1]$. In particular, for $p^{k'}$, contradicting to (A34). Thus, m_1^k must be undominated in the Bayesian bargaining problem Γ^k. Moreover, $\lim_{k\to\infty} U_1^k(p|m_1^k) = p - c_1 \leq U_1^*(p|m_1)$ for all $p \in [0,1]$. On the other hand, for each $k \in \mathbb{N}$, for any incentive feasible protocol $\tilde{m}$ in the Bayesian bargaining problem Γ^k, by (A33),

$$U^*_{2,\tilde{m}} = 1 - U_1^*(1|\tilde{m}) - \int_0^1 v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF_1(p) + (1-\bar{\mu})\frac{1}{k},$$

where v is the associate sub-gradient of $U_1^*(\cdot|\tilde{m})$ as defined above. Therefore, the payoff for 2 under the unique strong solution for 2, U^k_{2,m_2^k} must be bounded above by the payoff under solution of the

[34] Since otherwise we can construct another protocol n with the same v but with $U^*_{2,n} = 1 - \mathbb{E}_{F_1}[p] - c_2$. (A33) then ensures that n still dominates m_1^k.

following problem:

$$V_2^k := \max_{U_1^*, v, \mu} 1 - U_1^*(1|\tilde{m}) - \int_0^1 v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF_1(p) + (1 - \bar{\mu})\frac{1}{k}$$
$$\text{s.t. } v \text{ is non-decreasing,}$$
$$U_1^* \geq 1 - c_1.$$

Since as $k \to \infty$, $\{V_2^k\} \to U_{2,m_2}^*$, we also have

$$\limsup_{k\to\infty} U_{2,m_2^k}^k \leq \limsup_{k\to\infty} V_2^k = U_{2,m_2}^*.$$

Furthermore, for each $k \in \mathbb{N}$, denote U_1^k, v^k, μ^k as the optimal solutions. It then follows that $\lim_{k\to\infty} v^k(p) = w_2(p)$ and $U_1^k = 1 - c_1$ for all $k \in \mathbb{N}$. Therefore,

$$\lim_{k\to\infty} U_1^k(p|m_2^k) = U_1^*(p|m_2).$$

Similarly, if $\mathbb{E}_{F_1}[p] + c_2 \geq 1 - c_1$, we can then set $\delta := \mathbb{E}_{F_1}[p] + c_2 - (1 - c_1)$ and let $m_1^k(\{e\}|p) = 1$ for all p. By Lemma 5, $p - c_1 + \delta + \frac{1}{k}$ is undominated by any $m \in \mathcal{M}_4$. Therefore, by the same arguments, m_1^k is undominated and $\lim_{k\to\infty} U_1^k(p|m_1^k) = p - c_1 + \delta \leq \mathbb{E}_{F_1}[p] + c_2 = U_1^*(p|\bar{m})$ for all $p \in [0,1]$. Also, again by (A33), for each $k \in \mathbb{N}$, payoff for 2 under the unique strong solution for 2, $U_{2,m_2^k}^k$ will also be bounded above by the value the the following problem:

$$\begin{aligned} V_2^k := \max_{U_1^*, v, \mu} & 1 - U_1^*(1|\tilde{m}) - \int_0^1 v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF_1(p) + (1 - \bar{\mu})\left(\frac{1}{k} + \delta\right) \\ \text{s.t. } & v \text{ is non-decreasing,} \\ & U_1^* \geq 1 - c_1, \\ & U_1^* \geq v(p)(p - c_1) + (1 - \mu(p))\left(\frac{1}{k} + \delta\right) - \int_p^1 v(x)dx, \forall p \in [0,1], \end{aligned} \tag{A35}$$

where the last constraint follows from the characterization of (A31) and the boundedness of t, as argued in the proof of Lemma 4. Moreover, since v is increasing, it must be that in the optimal solution of (A35), $U_1^* \geq v(1)(1 - c_1) + (1 - \mu(p))(\frac{1}{k} + \delta)$ for all $p \in [0,1]$ and therefore, $U_1^* \geq v(1)(1 - c_1) + (1 - \bar{\mu})(\frac{1}{k} + \delta)$. Thus, the value of (A35), V_2^k must has:

$$V_2^k \leq \max_{v \in \mathcal{W}} \left[1 - v(1)(1 - c_1) - \int_0^1 v(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF_1(p)\right], \tag{A36}$$

where $\mathcal{W}$ is the set of non-decreasing functions from $[0,1]$ to $[0,1]$. Notice that the objective on the right hand side is exactly the problem for 2 under the original Bayesian bargaining problem Γ

if $v(1)$ is set to be 1. Similar to the proof of Lemma 5, the right hand side of (A36) must attain maximum at some extreme point of $\mathcal{W}$ and hence $v(1) \in \{0,1\}$, which then implies that

$$V_2^k \leq c_1 - \int_0^1 w_2(p)\left(c_1 + c_2 - \frac{F_1(p)}{f_1(p)}\right) dF_1(p) = U^*_{2,m_2},$$

for all $k \in \mathbb{N}$. Therefore,

$$\limsup_{k\to\infty} U^k_{2,m_2^k} \leq \limsup_{k\to\infty} V_2^k \leq U^*_{2,m_2}.$$

In fact, since the upper bound for V_2^k in (A36) is attainable be setting $U_1 = 1-c_1$ and $v = w_2$ in the problem (A35), for each $k \in \mathbb{N}$, the solution for (A35), (U_1^k, v^k, μ^k) must be such that $v^k(p) = w_2(k)$ and $U_1^k = 1 - c_1$. Therefore,

$$\lim_{k\to\infty} U_1^k(p|m_2^k) = U_1^*(p|m_2).$$

Finally, since in both cases, $u_2(e,p) = 1 - p - c_2$ for all $p \in [0,1]$,

$$U^k_{2,m_1^k} = 1 - \mathbb{E}_{F_1}[p] - c_2 = U^*_{2,m_1},$$

for all $k \in \mathbb{N}$.

Therefore, since m_1^k, m_2^k are strong solutions for 1 and 2 in the Bayesian bargaining problem Γ^k, respectively and since any convex combination of interim efficient protocols are interim efficient by Lemma B3, for any solution concept that satisfies the three axioms with allocation of bargaining power $(\phi, 1-\phi)$, $m^{k,\phi} := \phi m_1^k + (1-\phi)m_2^k$ is interim efficient and hence $m^{k,\phi} \in S(\Gamma^k)$.

Furthermore, since payoffs are linear in $\tilde{m}$ by the characterization, whenever $\tilde{m}$ is incentive feasible,

$$\begin{aligned}\limsup_{k\to\infty} U_1^k(p|m^{k,\phi}) &= \phi \limsup_{k\to\infty} U_1^*(p|m_1^k) + (1-\phi)\limsup_{k\to\infty} U_1^*(p|m_2^k)\\ &\leq \phi U_1^*(p|m_1) + (1-\phi)U_1^*(p|m_2)\\ &= U_1^*(p|m^\phi),\end{aligned}$$

for all $p \in [0,1]$ and

$$\begin{aligned}\limsup_{k\to\infty} U^k_{2,m^{k,\phi}} &= \phi \limsup_{k\to\infty} U^k_{2,m_1^k} + (1-\phi)\limsup_{k\to\infty} U^k_{2,m_2^k}\\ &\leq \phi U^*_{2,m_1} + (1-\phi)U^*_{2,m_2}\\ &= U^*_{2,m^\phi}.\end{aligned}$$

Therefore, for any solution concept that satisfies that three axioms, since m_1 and m_2 are interim efficient and thus m^ϕ is interim efficient by Lemma B3, the extension axiom then implies that $m^\phi \in S(\Gamma)$. Since this is true for any such S, $m^\phi \in \bar{S}^\phi(\Gamma)$. ■

B. Efficiency

In this section of the appendix, we will discuss the efficiency of protocols in various environments. In particular, we will characterize the set of ex-ante and ex-post efficient protocols without incentive feasibility constraints (i.e. the *first best* protocols) in each environment and discuss more about the efficient frontiers of the solutions we have obtained under the incentive feasibility constraints.

Consider first the environment of private value and one-sided incomplete information. For simplicity, we restrict the discussions to the case where only $c_1 \sim F_1$ is private information. From the fact that breakdowns are always costly to both parties, in particular, equations (1) and (2), it is clear that a protocol (w, t) is ex-post efficient if and only if $w \equiv 0$. Moreover, since ex-ante payoffs are linear in protocols m by Lemma 1 and the sum of ex-ante payoff is maximized if and only if $w \equiv 0$, according to (3), a protocol (w, t) is ex-ante efficient if and only if $w \equiv 0$. As shown in Proposition 2, there exists an incentive feasible protocol m^* that can achieve such unconstrained ex-ante and ex-post efficiency and is the essentially unique strong solution for player 1. However, when considering only the set of incentive feasible protocols and using the interim criterion, it is less obvious how the interim efficient frontier may behave since this requires comparing pointwisely the set of convex functions given in Lemma 1. Lemma B1 below shows that the frontier is a convex set. It then follows from the interim efficiency of strong solutions for both individuals that under the interim criterion, the set of efficient protocols in the set of incentive feasible protocols enlarges comparing to the unconstrianed case, since the strong solution for the party with incomplete information involves non-zero probability of breakdowns. This reflects some of the inefficiency results we have obtained.

Lemma B1. *Suppose that only c_1 is private information with $c_1 \sim F_1$ with density f_1. Let $m_0 = (w_0, t_0)$ and $m_1 = (w_1, t_1)$ be any two interim efficient protocols. Then for any $\phi \in [0, 1]$, $\phi m_0 + (1 - \phi)m_1$ is also interim efficient.*

Proof. Let $m^\phi := \phi m_0 + (1 - \phi)m_1$. Suppose that $\tilde{m} = (\tilde{w}, \tilde{t})$ dominates m^ϕ. Then $U_1^*(c_1|\tilde{m}) \geq U_1^*(c_1|m^\phi)$ for all $c_1 \in [\underline{c_1}, \overline{c_1}]$ and $U^*_{2,\tilde{m}} \geq U^*_{2,m^\phi}$ and at least one of the inequality is strict. From Lemma 1, we have:

$$
\begin{aligned}
U_i^*(c_1|\tilde{m}) = U_1^*(\underline{c_1}|\tilde{m}) - \int_{\underline{c_1}}^{c_1} \tilde{w}(x)dx \\
\geq U_1^*(\underline{c_1}|m^\phi) - \int_{\underline{c_1}}^{c_1} w^\phi(x)dx = U_1^*(c_1|m^\phi),
\end{aligned}
$$

for all $c_1 \in [\underline{c_1}, \overline{c_1}]$. And

$$U_1^*(\underline{c_1}|m^\phi) + U_{2,m^\phi}^* = 1 - \mathbb{E}\left[w^\phi(c_1)\left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)}\right)\right]$$
$$U_1^*(\underline{c_1}|\tilde{m}) + U_{2,\tilde{m}}^* = 1 - \mathbb{E}\left[\tilde{w}(c_1)\left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)}\right)\right].$$

Now notice that for such non-increasing function $\tilde{w}$ with $\tilde{w}(c_1) = 0$ whenever $c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} > 0$, for each $j \in \{0,1\}$, there exists $c_1^j \in [\underline{c_1}, \overline{c_1}]$ such that

$$-\int_{\underline{c_1}}^{c_1^j}(w_j(x) - \tilde{w}(x))dx > \int_{\underline{c_1}}^{\overline{c_1}}(w_j(x) - \tilde{w}(x))\left(x + c_2 - \frac{1-F_1(x)}{f_1(x)}\right)dF_1(x).$$

Indeed, by Lemma 1, we can define another protocol $\bar{m}$ by $(U_1^*(\underline{c_1}|\bar{m}), U_{2,\bar{m}}^*, \tilde{w})$, where

$$U_{2,\bar{m}}^* := U_{2,m_j}^*,$$
$$U_1(\underline{c_1}|\bar{m}) := 1 - U_{2,m_j}^* - \mathbb{E}\left[\tilde{w}(c_1)\left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)}\right)\right].$$

If $U_1(\underline{c_1}|\bar{m}) < p - \underline{c_1}$, since $U_1^*(\underline{c_1}|m_j) \geq p - \underline{c_1}$ by individual rationality, there exists c_1^j such that

$$U_1^*(\underline{c_1}|m_j) - \int_{\underline{c_1}}^{c_1^j} w_j(x)dx > U_1^*(\underline{c_1}|\bar{m}) - \int_{\underline{c_1}}^{c_1^j} \bar{w}(x)dx.$$

On the other hand, if $U_1^*(\underline{c_1}|\bar{m}) \geq p - \underline{c_1}$, then $\bar{m}$ is incentive feasible. As m_j is undominated, and $U_{2,m_j}^* = U_{2,\bar{m}}^*$, there must also exists $c_1^j \in [\underline{c_1}, \overline{c_1}]$ such that

$$U_1^*(\underline{c_1}|m_j) - \int_{\underline{c_1}}^{c_1^j} w_j(x)dx > U_1^*(\underline{c_1}|\bar{m}) - \int_{\underline{c_1}}^{c_1^j} \tilde{w}(x)dx.$$

Together, by condition 4 in Lemma 1 and by construction, $U_1^*(\underline{c_1}|m_j)$ and $U_1^*(\underline{c_1}|\bar{m})$ can be rewritten and therefore the above inequality becomes

$$-\int_{\underline{c_1}}^{c_1^j}(w_j(x) - \tilde{w}(x))dx > \int_{\underline{c_1}}^{\overline{c_1}}(w_j(x) - \tilde{w}(x))\left(x + c_2 - \frac{1-F_1(x)}{f_1(x)}\right)dF_1(x), \forall j \in \{0,1\}. \quad \text{(B1)}$$

Let $\tilde{c_1} := c_1^{j^*}$, where

$$j^* \in \underset{j\in\{0,1\}}{\operatorname{argmax}}\left\{-\int_{\underline{c_1}}^{c_1^j}(w_j(x) - \tilde{w}(x))dx\right\}.$$

On the other hand, since m^ϕ is dominated by $\tilde{m}$, as above, it must be that $U_1^*(c_1|m^\phi) + U_{2,m^\phi}^* \leq U_1^*(c_1|\tilde{m}) + U_{2,\tilde{m}^*}$ for all $c_1 \in [\underline{c_1}, \overline{c_1}]$. Using Lemma 1, this can be rewritten into:

$$1 - \mathbb{E}\left[w^\phi(c_1)\left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)}\right)\right] - \int_{\underline{c_1}}^{c_1} w^\phi(x)dx$$
$$\leq 1 - \mathbb{E}\left[\tilde{w}(c_1)\left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)}\right)\right] - \int_{\underline{c_1}}^{c_1} \tilde{w}(x)dx,$$

for all $c_1 \in [\underline{c_1}, \overline{c_1}]$. Rearranging, we have, for any $c_1 \in [\underline{c_1}, \overline{c_1}]$.

$$
\begin{aligned}
&-\int_{\underline{c_1}}^{c_1} (w^\phi(x) - \tilde{w}(x))dx \leq \int_{\underline{c_1}}^{\overline{c_1}} (w^\phi(x) - \tilde{w}(x)) \left(x + c_2 - \frac{1 - F_1(x)}{f_1(x)} \right) dF_1(x) \\
\Leftrightarrow &-\phi \int_{\underline{c_1}}^{c_1} (w_0(x) - \tilde{w}(x))dx - (1-\phi) \int_{\underline{c_1}}^{c_1} (w_1(x) - \tilde{w}(x))dx \\
\leq &\phi \int_{\underline{c_1}}^{\overline{c_1}} (w_0(x) - \tilde{w}(x)) \left(x + c_2 - \frac{1 - F_1(x)}{f_1(x)} \right) dF_1(x) + \\
&(1-\phi) \int_{\underline{c_1}}^{\overline{c_1}} (w_1(x) - \tilde{w}(x)) \left(x + c_2 - \frac{1 - F_1(x)}{f_1(x)} \right) dF_1(x).
\end{aligned}
$$

In particular, this holds for for $\tilde{c_1} \in [\underline{c_1}, \overline{c_1}]$. However, from (B1) and the definition of $\tilde{c_1}$,

$$
\begin{aligned}
&-\phi \int_{\underline{c_1}}^{\tilde{c_1}} (w_0(x) - \tilde{w}(x))dx - (1-\phi) \int_{\underline{c_1}}^{\tilde{c_1}} (w_1(x) - \tilde{w}(x))dx \\
\geq &-\phi \int_{\underline{c_1}}^{c_1^0} (w_0(x) - \tilde{w}(x))dx - (1-\phi) \int_{\underline{c_1}}^{c_1^1} (w_1(x) - \tilde{w}(x))dx \\
> &\phi \int_{\underline{c_1}}^{\overline{c_1}} (w_0(x) - \tilde{w}(x)) \left(x + c_2 - \frac{1 - F_1(x)}{f_1(x)} \right) dF_1(x) + \\
&(1-\phi) \int_{\underline{c_1}}^{\overline{c_1}} (w_1(x) - \tilde{w}(x)) \left(x + c_2 - \frac{1 - F_1(x)}{f_1(x)} \right) dF_1(x),
\end{aligned}
$$

a contradiction. Therefore, m^ϕ cannot be dominated. ■

Second, in the environment of private value with two-sided incomplete information, by the same reason, from (9) and (10), it follows directly that a protocol (w, t) is (unconstrained) ex-post efficient if and only if $w \equiv 0$. Also, from linearity of ex-ante payoffs and (11), a protocol (w, t) is (unconstrained) ex-ante efficient if and only if $w \equiv 0$. From Lemma 3, there exists an incentive feasible protocol that can achieve such ex-ante and ex-post unconstrained efficiency but cannot be selected under any allocation of bargaining powers. As in the case of one-sided incomplete information. The set of interim efficient protocols in the set of incentive feasible protocols contains more protocols than the set of unconstrained ex-ante and ex-post efficient ones. Specifically, as shown in Lemma B2, the interim efficient frontier is convex.

Lemma B2. *Suppose that only c_1 and c_2 are private information with $c_1 \sim F_1$ and $c_2 \sim F_2$ with densities f_1 and f_2, respectively. Let $m_0 = (w_0, t_0)$ and $m_1 = (w_1, t_1)$ be two interim efficient protocols. Then for any $\phi \in [0, 1]$, $\phi m_0 + (1-\phi)m_1$ is also interim efficient.*

Proof. For any $\phi \in [0, 1]$, let $m^\phi := \phi m_0 + (1-\phi)m_1$. Suppose that $\tilde{m} = (\tilde{w}, \tilde{t})$ dominates m^ϕ. We first claim that for each $j \in \{0,1\}$, there exists $c_1^j \in [\underline{c_1}, \overline{c_1}]$ and $c_2^j \in [\underline{c_2}, \overline{c_2}]$ such that:

$$
\begin{aligned}
&- \int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1^j} (w_j(x,c_2) - \tilde{w}(x,c_2)) dx dF_2(c_2) - \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2^j} (w_j(c_1,x) - \tilde{w}(c_1,x)) dx dF_1(c_1) \\
&> \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{\overline{c_2}} (w_j(c_1,c_2) - \tilde{w}(c_1,c_2)) \left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} - \frac{1-F_2(c_2)}{f_2(c_2)} \right) dF_1(c_1) dF_2(c_2).
\end{aligned}
$$

Indeed, for each $j \in \{0,1\}$, we can define another protocol $\hat{m}$ by $(U_1^*(\underline{c_1}|\hat{m}), U_2^*(\underline{c_2}|\hat{m}), \tilde{w})$, where

$$
U_2^*(\underline{c_2}|\hat{m}) := U_2^*(\underline{c_2}|m_j) - \min_{c_2 \in [\underline{c_2}, \overline{c_2}]} \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2} (w_j(c_1,x) - \tilde{w}(c_1,x)) dx dF_1(c_1)
$$

and

$$
U_1^*(\underline{c_1}|\hat{m}) := 1 - U_2(\underline{c_2}|\hat{m}) - \mathbb{E}\left[\tilde{w}(c_1,c_2) \left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} - \frac{1-F_2(c_2)}{f_2(c_2)} \right) \right].
$$

Notice that by such construction,

$$
\begin{aligned}
U_2^*(c_2|\hat{m}) &= U_2^*(\underline{c_2}|\hat{m}) - \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2} \tilde{w}(c_1,x) dx dF_1(c_1) \\
&\geq U_2^*(\underline{c_2}|m_j) - \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2} w_j(c_1,x) dx dF_1(c_1) = U_2^*(c_2|m_j),
\end{aligned}
$$

for all $c_2 \in [\underline{c_2}, \overline{c_2}]$. If $U_1^*(\underline{c_1}|\hat{m}) < p - \underline{c_1}$, then there exists c_1^j such that

$$
U_1^*(\underline{c_1}|\hat{m}) - \int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1^j} \tilde{w}(x,c_2) dx dF_2(c_2) < U_1^*(\underline{c_1}|m_j) - \int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1^j} w_j(x,c_2) dx dF_2(c_2),
$$

since m_j is individually rational for 1. If, on the other hand, $U_1^*(\underline{c_1}|\hat{m}) \geq p - \underline{c_1}$, then since $0 \leq \tilde{w} \leq 1$, and since m_j is incentive feasible, $\hat{m}$ must be incentive feasible. Since m_j is undominated and $U_2^*(c_2|\hat{m}) \geq U_2^*(c_2|m_j)$ for any $c_2 \in [\underline{c_2}, \overline{c_2}]$, there must exists some c_1^j such that

$$
U_1^*(\underline{c_1}|\hat{m}) - \int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1^j} \tilde{w}(x,c_2) dx dF_2(c_2) < U_1^*(\underline{c_1}|m_j) - \int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1^j} w_j(x,c_2) dx dF_2(c_2). \tag{B2}
$$

Take any

$$
c_2^j \in \operatorname*{argmin}_{c_2 \in [\underline{c_2}, \overline{c_2}]} \left[- \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2} (w_j(c_1,x) - \tilde{w}(c_1,x)) dx dF_1(c_1) \right].
$$

Then by construction of $U_1^*(\underline{c_1}|\hat{m})$ and the characterization for m_j given by Lemma 3, (B2) can be rewritten as

$$
\begin{aligned}
&- \int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1^j} (w_j(x,c_2) - \tilde{w}(x,c_2)) dx dF_2(c_2) - \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2^j} (w_j(c_1,x) - \tilde{w}(c_1,x)) dx dF_1(c_1) \\
&> \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{\overline{c_2}} (w_j(c_1,c_2) - \tilde{w}(c_1,c_2)) \left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} - \frac{1-F_2(c_2)}{f_2(c_2)} \right) dF_1(c_1) dF_2(c_2).
\end{aligned} \tag{B3}
$$

Let $\tilde{c_1} := c_1^{j^*}$ and $\tilde{c_2} := c_2^{j^*}$, where

$$j^* \in \underset{j\in\{0,1\}}{\operatorname{argmax}} \left\{ -\int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1^j} (w_j(x,c_2) - \tilde{w}(x,c_2)) dx dF_2(c_2) - \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2^j} (w_j(c_1,x) - \tilde{w}(c_1,x)) dx dF_1(c_1) \right\}.$$

Since $\tilde{m}$ dominates m^ϕ, $U_1^*(c_1|m^\phi) + U_2^*(c_2|m^\phi) \le U_1^*(c_1|\tilde{m}) + U_2^*(c_2|\tilde{m})$, for all $c_1 \in [\underline{c_1}, \overline{c_1}]$ and $c_2 \in [\underline{c_2}, \overline{c_2}]$. Again, since both m^ϕ and $\tilde{m}$ are incentive feasible, by Lemma 3,

$$\begin{aligned} &U_1^*(\underline{c_1}|m^\phi) + U_2^*(\underline{c_2}|m^\phi) - \int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1} w^\phi(x,c_2) dx dF_2(c_2) - \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2} w^\phi(c_1,x) dx dF_1(c_1) \\ \le &U_1^*(\underline{c_1}|\tilde{m}) + U_2^*(\underline{c_2}|\tilde{m}) - \int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1} \tilde{w}(x,c_2) dx dF_2(c_2) - \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2} \tilde{w}(c_1,x) dx dF_1(c_1), \end{aligned}$$

for all $c_1 \in [\underline{c_1}, \overline{c_1}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$ and

$$U_1^*(\underline{c_1}|m^\phi) + U_2^*(\underline{c_2}|m^\phi) = 1 - \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{\overline{c_2}} w^\phi(c_1,c_2) \left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} - \frac{1-F_2(c_2)}{f_2(c_2)} \right) dF_1(c_1) dF_2(c_2)$$

$$U_1^*(\underline{c_1}|\tilde{m}) + U_2^*(\underline{c_2}|\tilde{m}) = 1 - \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{\overline{c_2}} \tilde{w}(c_1,c_2) \left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} - \frac{1-F_2(c_2)}{f_2(c_2)} \right) dF_1(c_1) dF_2(c_2).$$

Together, since

$$U_i^*(\underline{c_i}|m^\phi) = \phi U_i^*(\underline{c_i}|m_0) + (1-\phi) U_i^*(\underline{c_i}|m_1),$$

for each $i \in \{1,2\}$ and

$$w^\phi \equiv \phi w_0 + (1-\phi) w_1,$$

for any $c_1 \in [\underline{c_1}, \overline{c_2}]$, $c_2 \in [\underline{c_2}, \overline{c_2}]$ we have:

$$\begin{aligned} &-\phi \left(\int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1} (w_0(x,c_2) - \tilde{w}(c_1,x)) dx dF_2(c_2) + \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2} (w_0(c_1,x) - \tilde{w}(c_1,x)) dx dF_1(c_1) \right) \\ &-(1-\phi) \left(\int_{\underline{c_2}}^{\overline{c_2}} \int_{\underline{c_1}}^{c_1} (w_1(x,c_2) - \tilde{w}(c_1,x)) dx dF_2(c_2) + \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{c_2} (w_1(c_1,x) - \tilde{w}(c_1,x)) dx dF_1(c_1) \right) \\ &\le \\ &\phi \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{\overline{c_2}} (w_0(c_1,c_2) - \tilde{w}(c_1,c_2)) \left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} - \frac{1-F_2(c_2)}{f_2(c_2)} \right) dF_1(c_1) dF_2(c_2) \\ &+(1-\phi) \int_{\underline{c_1}}^{\overline{c_1}} \int_{\underline{c_2}}^{\overline{c_2}} (w_1(c_1,c_2) - \tilde{w}(c_1,c_2)) \left(c_1 + c_2 - \frac{1-F_1(c_1)}{f_1(c_1)} - \frac{1-F_2(c_2)}{f_2(c_2)} \right) dF_1(c_1) dF_2(c_2). \end{aligned}$$

In particular, this holds for $c_1 = \tilde{c_1}$ and $c_2 = \tilde{c_2}$. However, by definition of $\tilde{c_1}$ and $\tilde{c_2}$ and from (B3), we have

$$
\begin{aligned}
&-\phi\left(\int_{\underline{c_2}}^{\overline{c_2}}\int_{\underline{c_1}}^{\tilde{c_1}}(w_0(x,c_2)-\tilde{w}(c_1,x))dxdF_2(c_2)+\int_{\underline{c_1}}^{\overline{c_1}}\int_{\underline{c_2}}^{\tilde{c_2}}(w_0(c_1,x)-\tilde{w}(c_1,x))dxdF_1(c_1)\right)\\
&-(1-\phi)\left(\int_{\underline{c_2}}^{\overline{c_2}}\int_{\underline{c_1}}^{\tilde{c_1}}(w_1(x,c_2)-\tilde{w}(c_1,x))dxdF_2(c_2)+\int_{\underline{c_1}}^{\overline{c_1}}\int_{\underline{c_2}}^{\tilde{c_2}}(w_1(c_1,x)-\tilde{w}(c_1,x))dxdF_1(c_1)\right)\\
&>\\
&\phi\int_{\underline{c_1}}^{\overline{c_1}}\int_{\underline{c_2}}^{\overline{c_2}}(w_0(c_1,c_2)-\tilde{w}(c_1,c_2))\left(c_1+c_2-\frac{1-F_1(c_1)}{f_1(c_1)}-\frac{1-F_2(c_2)}{f_2(c_2)}\right)dF_1(c_1)dF_2(c_2)\\
&+(1-\phi)\int_{\underline{c_1}}^{\overline{c_1}}\int_{\underline{c_2}}^{\overline{c_2}}(w_1(c_1,c_2)-\tilde{w}(c_1,c_2))\left(c_1+c_2-\frac{1-F_1(c_1)}{f_1(c_1)}-\frac{1-F_2(c_2)}{f_2(c_2)}\right)dF_1(c_1)dF_2(c_2),
\end{aligned}
$$

a contradiction. Therefore, m^ϕ cannot be dominated. ■

Analogously, in common value environments, the same arguments lead to the conclusion that a protocol is ex-post and ex-ante efficient if and only if $w \equiv 0$ for both one sided and two sided private information cases. However, this might not be incentive feasible if $c_1 + c_2 < \bar{p_1}(1) - \bar{p_2}(1)$. Furthermore, by exactly the same steps in the proofs of Lemma B1 and Lemma B2, and by noticing that, as shown in the proof of Proposition 7 and Proposition 8, for any incentive compatible protocol m, m is individually rational for 1 if and only if $U_1^*(1|m) \geq \bar{p} - c_1$, we can also see that the interim efficient frontier in common value environments is convex. This is summarized in the following two lemmas.[35]

Lemma B3. *Suppose that $p \sim F_1$ with density f_1 and that only 1 can observe its realization. Let m_0 and m_1 be two interim efficient protocols. Then for any $\phi \in [0,1]$, $\phi m_0 + (1-\phi)m_1$ is also interim efficient.*

Lemma B4. *Suppose that $\theta_1 \sim G_1$, $\theta_2 \sim G_2$, that only 1 can observe realizations of θ_1 and only 2 can observe realizations of θ_2, and that Assumption 5 holds. Let m_0 and m_1 be two interim efficient protocols. Then for any $\phi \in [0,1]$, $\phi m_0 + (1-\phi)m_1$ is also interim efficient.*

[35]The proofs of these two lemmas are completely analogous to the proofs of Lemma B1 and Lemma B2 and therefore are omitted.