A Note on Topological Properties of Outcomes in a Monopoly Market

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Abstract

A monopolist with a nonnegative constant marginal cost faces an arbitrary nondecreasing and upper-semicontinuous demand function on $\mathbb{R}_+$ that takes a value in $\{0, 1\}$ outside of a fixed compact interval. This note derives topological properties of outcomes induced by this monopolist's optimal pricing problem. Specifically, the monopolist's optimal profit is continuous in both the marginal cost and the demand (under the weak-* topology); the induced output is lower (upper)-semicontinuous in both the marginal cost and the demand when the monopolist always charges the highest (lowest) optimal price; the optimal price correspondence is upper-hemicontinuous in both the marginal cost and the demand, which in turn implies that the consumer surplus is upper (lower)-semicontinuous in both the marginal cost and the demand when the monopolist always charges the lowest (highest) optimal price. These results further imply similar topological properties of outcomes in settings that feature either second-degree price discrimination or third-degree price discrimination.

KEYWORDS: Topological properties, monopoly pricing, continuity, second-degree price discrimination, third-degree price discrimination.

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1 Introduction

For more than a century, monopolistic pricing has been an ubiquitous topic in economics. The cononical monopoly pricing problem has countless applications across various fields of economics, including industrial organization, labor economics, law and economics, informational economics, and mechanism design. Among these applications, an emerging literature in recent years has been studying the monopoly pricing problem where the demand and cost structure are either endogenous, or are part of the comparative statics analysis (see, for instance, Johnson and Myatt (2006), Bergemann et al. (2015), Roesler and Szentes (2017), Ravid et al. (2019), Condorelli and Szentes (2020), Yang (2019), Yang (2020)). As a result, topological properties of outcomes (e.g., firm profit, output, consumer surplus) in a monopoly pricing setting has become methodologically more relevant. This note outlines the topological properties of these outcomes when regarded as functions of the demand function and the monopolist's cost structure.

Specifically, in this note, I show that the monopolist's optimal profit is continuous in both the marginal cost and the demand (under the weak-* topology); the induced output is lower (upper)-semicontinuous in both the marginal cost and the demand when the monopolist always charges the highest (lowest) optimal price; the optimal price correspondence is upper-hemicontinuous in both the marginal cost and the demand, which in turn implies that the consumer surplus is upper (lower)-semicontinuous in both the marginal cost and the demand when the monopolist always charges the lowest (highest) optimal price.

Furthermore, I apply these results and further derive similar topological properties of outcomes in settings that feature either second-degree price discrimination or third-degree price discrimination. In an environment that features second-degree price discrimination, I show that the aforementioned continuity properties, when regarding outcomes as functions of the demand and the cost structure, still hold. Meanwhile, in an environment that features third-degree price discrimination, I show that these properties, when when regarding outcomes as functions of the underlying market segmentations, continue to hold as well.

The rest of this note is organized as follows. Section 2 introduce the notation and set up; Section 3 outlines the continuity results in a monopoly pricing setting when the monopolist has constant marginal costs; Section 4 then applies these results and derive similar properties in settings that feature second-degree price discrimination and third-degree price discrimination. Section 5 concludes.

2 Model

Let $V := [\underline{v}, \overline{v}] \subset \mathbb{R}_+$ be a compact interval on $\mathbb{R}_+$ and let $\mathcal{D}$ be the collection nonincreasing and upper-semicontinuous functions $D : \mathbb{R}_+ \to [0, 1]$ such that $D(\underline{v}) = 1$ and $D(\overline{v}^+) = 0$.

Notice that $\mathcal{D}$ is isomorphic to the collection of probability measures on V (endowed with the Borel σ-algebra).[1] Therefore, $\mathcal{D}$ can be endowed with the weak-* topology, so that $\{D_n\} \to D$ if and only if for any bounded continuous function $h : \mathbb{R}_+ \to \mathbb{R}$,

$$\lim_{n\to\infty} \int_0^\infty h(v) D_n(\mathrm{d}v) = \int_0^\infty h(v) D(\mathrm{d}v),$$

where for any $\hat{D} \in \mathcal{D}$

$$\int_0^\infty h(v)\hat{D}(\mathrm{d}v) := m^{\hat{D}}(\mathrm{d}v)$$

and $m^{\hat{D}}$ is the (unique) probability measure in $\Delta(V)$ that is associated with $\hat{D}$.

A monopolist has a constant marginal cost $c \geq 0$ and faces a demand function $D \in \mathcal{D}$. Her optimal pricing problem is given by[2]

$$\max_{p\geq 0}(p - c)D(p). \tag{1}$$

For any $D \in \mathcal{D}$ and for any $c \geq 0$, let

$$\pi_D(c) := \max_{p\geq 0}(p - c)D(p)$$

be the monopolist's optimal profit and let

$$\boldsymbol{P}_D(c) := \operatorname*{argmax}_{p\in[0,\overline{v}+1]}(p - c)D(p).$$

be the set of optimal prices. By definition, $\boldsymbol{P} : \mathcal{D} \times \mathbb{R}_+ \rightrightarrows \mathbb{R}$ is a correspondence. A selection of $\boldsymbol{P}$ is denoted by $\boldsymbol{p}$ while $\overline{\boldsymbol{p}}_D(c) := \max \boldsymbol{P}_D(c)$ and $\underline{\boldsymbol{p}}_D(c) := \min \boldsymbol{P}_D(c)$ denote the highest and the lowest optimal price for the monopolist, respectively.

For any $D \in \mathcal{D}$, any $c \geq 0$ and for any $\boldsymbol{p} \in \boldsymbol{P}$, let

$$Q(D, c|\boldsymbol{p}) := D(\boldsymbol{p}_D(c))$$

denote the induced output when the demand is D, the marginal cost is c, and the optimal price is $\boldsymbol{p}_D(c)$. Meanwhile, let

$$\Sigma(D, c|\boldsymbol{p}) := \int_{\{v\geq \boldsymbol{p}_D(c)\}} (v - \boldsymbol{p}_D(c))D(\mathrm{d}v)$$

denote the induced consumer surplus when the demand is D, the marginal cost is c, and the optimal price is $\boldsymbol{p}_D(c)$.

[1] To see this, notice that $D : \mathbb{R}_+ \to [0, 1]$ being nonincreasing and upper-semicontinuous is equivalent to D being nonincreasing and left continuous, and hence $1 - D$ is a CDF with support on V.

[2] A solution must exist because V is compact and D is upper-semicontinuous.

3 Topological Properties of Monopoly Outcomes

This section outlines topological properties of the outcomes in the monopolistic pricing setting defined above.

Proposition 1. *For any $D \in \mathcal{D}$, $\pi_D : C \to \mathbb{R}_+$ is continuous and convex. Furthermore, for any $\boldsymbol{p} \in \boldsymbol{P}$, and for any $c \in C$, $-D(\boldsymbol{p}_D(c))$ is a subgradient of π_D at c. In particular, for any $c < c'$,*

$$\pi_D(c) - \pi_D(c') = \int_c^{c'} D(\boldsymbol{p}_D(z))\,\mathrm{d}z, \tag{2}$$

for any $\boldsymbol{p} \in \boldsymbol{P}$

Proof. By definition of π_D, for any $c \geq 0$,

$$\pi_D(c) = \max_{p \in \mathbb{R}_+} (p - c)D(p).$$

As such, π_D is convex for all $D \in \mathcal{D}$ since it is the pointwise supremum of a family of affine functions. Moreover, for any $\boldsymbol{p} \in \boldsymbol{P}$ and for any c, c',

$$\begin{aligned}
0 \leq & \pi_D(c') - (\boldsymbol{p}_D(c) - c')D(\boldsymbol{p}_D(c)) \\
= & \pi_D(c') + c'D(\boldsymbol{p}_D(c)) - \boldsymbol{p}_D(c)D(\boldsymbol{p}_D(c)) \\
= & \pi_D(c') - cD(\boldsymbol{p}_D(c)) + c'D(\boldsymbol{p}_D(c)) - (\boldsymbol{p}_D(c) - c)D(\boldsymbol{p}_D(c)) \\
= & \pi_D(c') - [-D(\boldsymbol{p}_D(c))(c' - c) + \pi_D(c)].
\end{aligned}$$

Thus, $-D(\boldsymbol{p}_D(c))$ is a subgradient of π_D at c. Together with convexity of π_D, π_D is differentiable almost everywhere and

$$\pi'_D(c) = -D(\boldsymbol{p}_D(c)),$$

for almost all $c \in C$. Thus, since π_D is convex, for any $c < c'$,

$$\pi_D(c) - \pi_D(c') = \int_c^{c'} D(\boldsymbol{p}_D(z))\,\mathrm{d}z,$$

for any $\boldsymbol{p} \in \boldsymbol{P}$.

For continuity, notice that for any $c \geq 0$,

$$\pi_D(c) = \max_{(p,q) \in \Xi} (p - c)q,$$

where $\Xi := \mathrm{cl}(\{(p, D(p)) : p \in V\})$ is a compact set in $\mathbb{R}^2$. Therefore, by Berge's theorem of maximum, π_D is continuous. ■

Proposition 2. *For any $D \in \mathcal{D}$, the correspondence $\boldsymbol{P}_D : C \rightrightarrows \mathbb{R}_+$ is upper-hemicontinuous. In particular, $\overline{\boldsymbol{p}}_D : C \to \mathbb{R}_+$ is upper-semicontinuous and $\underline{\boldsymbol{p}}_D : C \to \mathbb{R}_+$ is lower-semicontinuous.*

Proof. Consider any $D \in \mathcal{D}$. To show upper-hemicontinuity of $\boldsymbol{P}_D$, it suffices to show that for any sequences $\{c_n\} \subseteq C$ and $\{p_n\} \subseteq \mathbb{R}_+$ such that $\{p_n\} \to p \in \mathbb{R}_+$ and $\{c_n\} \to c \in \mathbb{R}_+$ and that $p_n \in \boldsymbol{P}_D(c_n)$ for all $n \in \mathbb{N}$, $p \in \boldsymbol{P}_D(c)$. Indeed, for any $n \in \mathbb{N}$, since $p_n \in \boldsymbol{P}_D(c_n)$, $\pi_D(c_n) = (p_n - c_n)D(p_n)$ for all $n \in \mathbb{N}$. Moreover, since $\pi_D : C \to \mathbb{R}_+$ according to Proposition 1,

$$\lim_{n\to\infty} \pi_D(c_n) = \pi_D(c).$$

Therefore, since D is upper-semicontinuous,

$$\pi_D(c) = \lim_{n\to\infty} \pi_D(c_n) = \limsup_{n\to\infty}(p_n - c_n)D(p_n) \leq (p - c)D(p) \leq \pi_D(c).$$

Thus, $p \in \boldsymbol{P}_D(c)$ as desired. Finally, since for any $\boldsymbol{p} \in \boldsymbol{P}$, $\boldsymbol{p}_D : C \to \mathbb{R}_+$ is nondecreasing, upper-hemicontinuity of $\boldsymbol{p}_D$ then implies right-continuity of $\overline{\boldsymbol{p}}_D$ and left-continuity of $\underline{\boldsymbol{p}}_D$. This completes the proof. ■

Proposition 3. *For any $D \in \mathcal{D}$, $Q(D, \cdot|\overline{\boldsymbol{p}}) : \mathbb{R}_+ \to [0, 1]$ is lower-semicontinuous and $Q(D, \cdot|\underline{\boldsymbol{p}}) : \mathbb{R}_+ \to [0, 1]$ is upper-semicontinuous.*

Proof. Consider any $D \in \mathcal{D}$ and any $c \geq 0$. By Proposition 2,

$$\lim_{c'\downarrow c} \overline{\boldsymbol{p}}_D(c') = \overline{\boldsymbol{p}}_D(c).$$

Together with continuity of π_D, which is due to Proposition 1,

$$\begin{aligned}(\overline{\boldsymbol{p}}_D(c) - c)D(\overline{\boldsymbol{p}}_D(c)) =& \pi_D(c) \\ =& \lim_{c'\downarrow c} \pi_D(c') \\ =& \lim_{c'\downarrow c}(\overline{\boldsymbol{p}}_D(c') - c')D(\overline{\boldsymbol{p}}_D(c')) \\ =& (\overline{\boldsymbol{p}}_D(c) - c) \cdot \lim_{c'\downarrow c} D(\overline{\boldsymbol{p}}_D(c')),\end{aligned}$$

and hence

$$\lim_{c'\downarrow c} D(\overline{\boldsymbol{p}}_D(c')) = D(\overline{\boldsymbol{p}}_D(c)).$$

Thus, $Q(D, \cdot|\overline{\boldsymbol{p}})$ is right-continuous, which is equivalent to lower-semicontinuity given that $Q(D, \cdot|\overline{\boldsymbol{p}})$ is nonincreasing, which in turn follows from Proposition 1. The proof of upper-semicontinuity of $Q(D, \cdot|\underline{\boldsymbol{p}})$ is analogous. This completes the proof. ■

Proposition 4. *For any $c \geq 0$, $D \to \pi_D(c)$ is continuous on $\mathcal{D}$.*

Proof. Since $V \subseteq \mathbb{R}_+$ is bounded, this lemma is a special case of Theorem 12 of Hart and Reny (2019) when $k = 1$. ■

Proposition 5. *For any $c > 0$, $Q(\cdot, c|\overline{\boldsymbol{p}}) : \mathcal{D} \to [0,1]$ is lower-semicontinuous and $Q(\cdot, c|\underline{\boldsymbol{p}}) : \mathcal{D} \to [0,1]$ is upper-semicontinuous.*

Proof. For any $c > 0$ and for any $D \in \mathcal{D}$, define $\pi'_D(c^+)$ as

$$\pi'_D(c^+) := \lim_{c' \downarrow c} \frac{\pi_D(c') - \pi_D(c)}{c' - c}.$$

Since π_D is convex, $\pi'_D(c^+)$ is well-defined. Furthermore, by Proposition 1, $-D(\overline{\boldsymbol{p}}_D(c))$ is a subgradient of π_D at c and therefore, for any $c' > c$,

$$\frac{\pi_D(c') - \pi_D(c)}{c' - c} \geq -D(\overline{\boldsymbol{p}}_D(c)),$$

which implies that

$$\pi'_D(c^+) \geq -D(\overline{\boldsymbol{p}}_D(c)). \tag{3}$$

Meanwhile, by (2), for any $c' > c$,

$$\frac{\pi_D(c') - \pi_D(c)}{c' - c} = \frac{1}{c' - c} \int_c^{c'} -D(\overline{\boldsymbol{p}}_D(z))\,\mathrm{d}z \leq -D(\overline{\boldsymbol{p}}_D(c')).$$

Thus, by Proposition 3,

$$\pi'_D(c^+) \leq \lim_{c' \downarrow c} -D(\overline{\boldsymbol{p}}_D(c')) = -D(\overline{\boldsymbol{p}}_D(c)). \tag{4}$$

Combining (3) and (4),

$$\pi'_D(c^+) = -D(\overline{\boldsymbol{p}}_D(c)).$$

Now consider any $D \in \mathcal{D}$ and any $\{D_n\} \subseteq \mathcal{D}$ such that $\{D_n\} \to D$, Proposition 4 implies that $\{\pi_{D_n}\} \to \pi_D$ pointwise. Thus, for any $c > 0$, by Theorem 24.5 of Rockafellar (1970),

$$-\liminf_{n\to\infty} D_n(\overline{\boldsymbol{p}}_{D_n}(c)) = \limsup_{n\to\infty} \pi'_{D_n}(c^+) \leq \pi'_D(c^+) = -D(\overline{\boldsymbol{p}}_D(c)).$$

Therefore, for any $c > 0$,

$$\liminf_{n\to\infty} D_n(\overline{\boldsymbol{p}}_{D_n}(c)) \geq D(\overline{\boldsymbol{p}}_D(c)),$$

as desired. The proof of upper-semicontinuity of $Q(\cdot, c|\underline{\boldsymbol{p}})$ is analogous. ■

Proposition 6. *For any $c \geq 0$, the correspondence $D \mapsto \boldsymbol{P}_D(c)$ is upper-hemicontinuous on $\mathcal{D}$.*

Proof. Consider any $c \geq 0$ and any sequence $\{D_n\} \subseteq \mathcal{D}$ such that $\{D_n\} \to D$ for some $D \in \mathcal{D}$. Let

$$\bar{p} := \limsup_{n\to\infty} \overline{\boldsymbol{p}}_{D_n}(c).$$

Take a subsequence $\{D_{n_k}\} \subseteq \{D_n\}$ such that

$$\lim_{k\to\infty} \overline{\boldsymbol{p}}_{D_{n_k}}(c) = \bar{p}.$$

First notice that since $D \in \mathcal{D}$ is upper-semicontinuous, for any sequence $\{\delta_k\} \subset \mathbb{R}_+$ such that $\{\delta_k\} \to 0$,

$$\limsup_{k\to\infty} (\overline{\boldsymbol{p}}_{D_{n_k}}(c) - c) D_{n_k}(\overline{\boldsymbol{p}}_{D_{n_k}}(c) - \delta_k) \leq (\bar{p} - c) D(\bar{p}). \tag{5}$$

Moreover, by the definition of the Lévy Prokhorov metric, for any $k \in \mathbb{N}$,

$$D_{n_k}(\overline{\boldsymbol{p}}_{D_{n_k}}(c)) \leq D\left(\overline{\boldsymbol{p}}_{D_{n_k}}(c) - \left(\rho(D_{n_k}, D) + \frac{1}{k}\right)\right) - \left(\rho(D_{n_k}, D) + \frac{1}{k}\right), \tag{6}$$

where $\rho : \mathcal{D} \times \mathcal{D} \to \mathbb{R}_+$ is the Lévy Prokhorov metric. Together, since $\{\rho(D_{n_k}, D)\} \to 0$ as $k \to \infty$, which is because $\{D_{n_k}\} \to D$ as $k \to \infty$, we have

$$\begin{aligned}
\pi_D(c) &= \lim_{k\to\infty} \pi_{D_{n_k}}(c) \\
&= \lim_{k\to\infty} (\overline{\boldsymbol{p}}_{D_{n_k}}(c) - c) D_{n_k}(\overline{\boldsymbol{p}}_{D_{n_k}}(c)) \\
&\leq \limsup_{k\to\infty} (\overline{\boldsymbol{p}}_{D_{n_k}}(c) - c) \left[D\left(\overline{\boldsymbol{p}}_{D_{n_k}}(c) - \left(\rho(D_{n_k}, D) + \frac{1}{k}\right)\right) - \left(\rho(D_{n_k}, D) + \frac{1}{k}\right)\right] \\
&\leq (\bar{p} - c) D(\bar{p}) \\
&\leq \pi_D(c),
\end{aligned}$$

where the first equality follows from Proposition 4, the first inequality follows from (6), and the second inequality follows from $\{\rho(D_{n_k}, D)\} \to 0$ as $k \to \infty$, (5), as well as the fact that $\overline{\boldsymbol{p}}_{D_{n_k}}(c) \leq \overline{v} + 1 < \infty$. As a result, it then follows that $\bar{p} \in \boldsymbol{P}_D(c)$ and therefore $\bar{p} \leq \overline{\boldsymbol{p}}_D(c)$. Thus,

$$\limsup_{n\to\infty} \overline{\boldsymbol{p}}_{D_n}(c) = \bar{p} \leq \overline{\boldsymbol{p}}_D(c).$$

Therefore, $D \mapsto \overline{\boldsymbol{p}}_D(c)$ is upper-semicontinuous. By similar arguments, $D \mapsto \underline{\boldsymbol{p}}_D(c)$ is lower-semicontinuous. Together, these imply that $D \mapsto \boldsymbol{P}_D(c)$ is upper-hemicontinuous. This completes the proof. ∎

Proposition 7. *For any $D \in \mathcal{D}$, $\Sigma(D, \cdot|\overline{\boldsymbol{p}}) : \mathbb{R}_+ \to \mathbb{R}_+$ is lower-semicontinuous and $\Sigma(D, \cdot|\underline{\boldsymbol{p}}) : \mathbb{R}_+ \to \mathbb{R}_+$ is upper-semicontinuous. Moreover, for any $c \geq 0$, $\Sigma(\cdot, c|\overline{\boldsymbol{p}}) : \mathcal{D} \to \mathbb{R}_+$ is lower-semicontinuous and $\Sigma(\cdot, c|\underline{\boldsymbol{p}}) : \mathcal{D} \to \mathbb{R}_+$ is upper-semicontinuous.*

Proof. Consider any $D \in \mathcal{D}$ and any $\{c_n\} \subset \mathbb{R}_+$ such that $\{c_n\} \to c$. By Proposition 2, for any $v \in V$,

$$\liminf_{n\to\infty} (v - \overline{\boldsymbol{p}}_D(c_n))^+ \geq \overline{\boldsymbol{p}}_D(c)$$

and

$$\limsup_{n\to\infty}(v - \underline{\boldsymbol{p}}_D(c_n))^+ \leq \underline{\boldsymbol{p}}_D(c).$$

Therefore, by Fatou's lemma and the reversed Fatou's lemma,

$$\liminf_{n\to\infty} \int_V (v - \overline{\boldsymbol{p}}_D(c_n))^+ D(\mathrm{d}v) \geq \int_V (v - \overline{\boldsymbol{p}}_D(c))^+ D(\mathrm{d}v)$$

and

$$\limsup_{n\to\infty} \int_V (v - \underline{\boldsymbol{p}}_D(c_n))^+ D(\mathrm{d}v) \leq \int_V (v - \underline{\boldsymbol{p}}_D(c))^+ D(\mathrm{d}v).$$

Furthermore, notice that for any $c \geq 0$ and for any $\boldsymbol{p} \in \boldsymbol{P}$, using integration parts,

$$\int_V (v - \boldsymbol{p}_D(c))^+ D(\mathrm{d}v) = \int_{\{v \geq \boldsymbol{p}_D(c)\}} D(v)\,\mathrm{d}v = \int_V \mathbf{1}\{v \geq \boldsymbol{p}_D(c)\} D(v)\,\mathrm{d}v. \tag{7}$$

Meanwhile, for any $c \geq 0$ and for any $\{D_n\} \subseteq \mathcal{D}$ such that $\{D_n\} \to D$, Proposition 6 implies that, for any $v \geq 0$

$$\liminf_{n\to\infty} \mathbf{1}\{v \geq \overline{\boldsymbol{p}}_{D_n}(c)\} \geq \mathbf{1}\{v \geq \overline{\boldsymbol{p}}_D(c)\}$$

and

$$\limsup_{n\to\infty} \mathbf{1}\{v \geq \underline{\boldsymbol{p}}_{D_n}(c)\} \leq \mathbf{1}\{v \geq \underline{\boldsymbol{p}}_D(c)\}.$$

Together, by Fatou's lemma and the reversed Fatou's lemma,

$$\liminf_{n\to\infty} \int_V \mathbf{1}\{v \geq \overline{\boldsymbol{p}}_{D_n}(c)\} D(v)\,\mathrm{d}v \geq \int_V \mathbf{1}\{v \geq \overline{\boldsymbol{p}}_D(c)\} D(v)\,\mathrm{d}v$$

and

$$\limsup_{n\to\infty} \int_V \mathbf{1}\{v \geq \underline{\boldsymbol{p}}_{D_n}(c)\} D(v)\,\mathrm{d}v \leq \int_V \mathbf{1}\{v \geq \underline{\boldsymbol{p}}_D(c)\} D(v)\,\mathrm{d}v.$$

This completes the proof. ■

4 Outcomes Induced by Price Discrimination

In this section, I apply the results above to further derive topological properties of outcomes in markets that feature second and third degree price discrimination.

4.1 Second-Degree Price Discrimination

Consider the following model that features second-degree price discrimination, which is due to Mussa and Rosen (1978). There is a unit mass of consumers with heterogeneous preferences for product qualities so that for a consumer with value $v \in V$, his utility of consuming the product with quality $q \in [0, 1]$ is vq. Furthermore, assume that the consumers' preferences are quasi-linear in money. Across the consumers, their values are distributed according to a probability measure and is summarized by some $D \in \mathcal{D}$ so that $D(p)$ is the share of consumers with values above p.

Product qualities are denoted by $q \in [0,1]$. To produce a product with quality q, it costs the monopolist $\boldsymbol{C}(q) = \int_0^q \boldsymbol{c}(x)\,\mathrm{d}x$, where $\boldsymbol{c} : \mathbb{R} \to \mathbb{R}_+$ is a nondecreasing (and hence, without loss, right-continuous) function. The monopolist can post any menu $\{(q,t)\}$ so that an item (q,t) denotes a product with quality q and costs t for the consumers. Let $\mathcal{C}$ denote the collection of nondecreasing and right-continuous functions from $\mathbb{R}$ to $\mathbb{R}_+$ such that $\boldsymbol{c}(0^-) = 0$, and endow this set with the weak-* topology.

For any $v \geq 0$, let

$$\boldsymbol{c}^{-1}(v) := \inf\{q \in [0,1] | \boldsymbol{c}(q) \geq v\}$$

and let $U(v|\boldsymbol{c}) := \max_{q\in[0,1]}[qv - \boldsymbol{C}(q)]$. Notice that $\boldsymbol{c}^{-1} \in \mathcal{C}$ for all $\boldsymbol{c} \in \mathcal{C}$; $\boldsymbol{c}^{-1}(v) \in \operatorname{argmax}_{q\in[0,1]}[qv - \boldsymbol{C}(q)]$ for all $v \geq 0$; and that $\{\boldsymbol{c}_n\} \to \boldsymbol{c}$ if and only if $\{\boldsymbol{c}_n^{-1}\} \to \boldsymbol{c}^{-1}$. Moreover, for any $D \in \mathcal{D}$ and for any $q \in [0,1]$, let $D^{-1}(q) := \sup\{p \geq 0 | D(p) \geq q\}$ and let $R_D(q) := qD^{-1}(q)$. Denote $\bar{R}_D$ as the concave closure of R_D, that is, $\bar{R}_D := \operatorname{cav}(R_D)$ and let $r_D(q) := \max \partial \bar{R}_D(q)$ be the largest subgradient of $\bar{R}_D$ at q, for all $q \in [0,1]$.[3]

Mussa and Rosen (1978) show[4] that the monopolist's optimal profit is

$$\Pi^{\mathrm{II}}(D, \boldsymbol{c}) := \int_V U(r_D(D(v))|\boldsymbol{c}) D(\mathrm{d}v)$$

while the consumer surplus is

$$\Sigma^{\mathrm{II}}(D, \boldsymbol{c}) := \int_V \boldsymbol{c}^{-1}(r_D(D(v))) D(v)\,\mathrm{d}v.$$

The next lemma connects the marginal revenue r_D with the optimal monopoly price given a certain marginal cost.

Lemma 1. *For any $c \geq 0$, let $\underline{\boldsymbol{p}}_D^{-1}(v) := \inf\{c \geq 0 | \underline{\boldsymbol{p}}_D(c) \geq v\}$. Then $\underline{\boldsymbol{p}}_D^{-1}(v) = r_D(D(v))$ for all $v \in V$.*

Proof. For any $c \in C$ and for any $D \in \mathcal{D}$, rewrite the monopolist's pricing problem (1) as

$$\max_{q\in[0,1]} [R_D(q) - cq]$$

and let $\boldsymbol{q}(c)$ be the largest solution. Since the largest element of $\operatorname{argmax}_{q\in[0,1]}[R_D(q) - cq]$) equals to the largest element of $\operatorname{argmax}_{q\in[0,1]}[\bar{R}_D(q) - cq]$, it must be that $\boldsymbol{q}(c) = \sup\{q \in [0,1] | r_D(q) \geq c\}$. Moreover, by definition, $\boldsymbol{q}_D(c) = D(\underline{\boldsymbol{p}}_D(c))$. Therefore, for all $v \in V$,

$$D(v) = D(\underline{\boldsymbol{p}}_D(\underline{\boldsymbol{p}}_D^{-1}(v))) = \boldsymbol{q}_D(\underline{\boldsymbol{p}}_D^{-1}(v)) = \sup\{q \in [0,1] | r_D(q) \geq \underline{\boldsymbol{p}}_D^{-1}(v)\},$$

[3] $r_D(D(v))$ is also known as the ironed virtual value, see Myerson (1981).

[4] Mussa and Rosen (1978) only derive results where D is absolutely continuous. In this case $r_D(D(v)) = v + D(v)/D'(v)$, where D' is the Radon-Nikodym derivative of D. This generalized version that admits any $D \in \mathcal{D}$ can be proved using analogous arguments.

and hence, as $D(v) = \sup\{q \in [0,1] | q \leq D(v)\} = \sup\{q \in [0,1] | r_D(q) \geq r_D(D(v))\}$, $r_D(D(v)) = \underline{\boldsymbol{p}}_D^{-1}(v)$. This completes the proof. ■

With (1), I now introduce the following crucial lemma.

Lemma 2. *For any* $\boldsymbol{c} \in \mathcal{C}$ *and for any* $D \in \mathcal{D}$,

$$\int_V U(r_D(D(v))|\boldsymbol{c})D(\mathrm{d}v) = \int_0^\infty \pi_D(c)\boldsymbol{c}^{-1}(\mathrm{d}c).$$

and

$$\int_V \boldsymbol{c}^{-1}(r_D(D(v)))D(v)\,\mathrm{d}v = \int_0^\infty \left(\int_V (v - \underline{\boldsymbol{p}}_D(c))^+ D(\mathrm{d}v)\right) \boldsymbol{c}^{-1}(\mathrm{d}c).$$

Proof. Notice that by the envelope theorem (Milgrom and Segal, 2002), for any $\boldsymbol{c} \in \mathcal{C}$, $U(\cdot|\boldsymbol{c})$ is absolutely continuous and its almost everywhere derivative at $v \geq 0$ equals to $\boldsymbol{c}^{-1}(v)$. By Lemma 1,

$$\int_V U(r_D(D(v))|\boldsymbol{c})D(\mathrm{d}v) = \int_0^\infty U(\underline{\boldsymbol{p}}_D^{-1}(v)|\boldsymbol{c})D(\mathrm{d}v) = \int_0^\infty U(c|\boldsymbol{c})D \circ \underline{\boldsymbol{p}}_D(\mathrm{d}c).$$

Furthermore, using integration parts, together with the fact that $U(0) = 0$,

$$\int_0^\infty U(c|\boldsymbol{c})D \circ \underline{\boldsymbol{p}}_D(\mathrm{d}c) = \int_0^\infty U'(c|\boldsymbol{c})D(\underline{\boldsymbol{p}}_D(c))\,\mathrm{d}c = \int_0^\infty \boldsymbol{c}^{-1}(c)D(\underline{\boldsymbol{p}}_D(c))\,\mathrm{d}c.$$

Together with Proposition 1, using integration by parts again, we have

$$\int_0^\infty \boldsymbol{c}^{-1}(c)D(\underline{\boldsymbol{p}}_D(c))\,\mathrm{d}c = \boldsymbol{c}^{-1}(0^-)\pi_D(0) + \int_0^\infty \pi_D(c)\boldsymbol{c}^{-1}(\mathrm{d}c) = \int_0^\infty \pi_D(c)\boldsymbol{c}^{-1}(\mathrm{d}c),$$

where the last equality is follows from $\boldsymbol{c}^{-1}(0^-) = 0$, which in turn is due to $\boldsymbol{c}^{-1} \in \mathcal{C}$. Meanwhile, by (7) and Lemma 1,

$$\begin{aligned}\int_0^\infty \left(\int_V (v - \underline{\boldsymbol{p}}_D(c))^+ D(\mathrm{d}v)\right) \boldsymbol{c}^{-1}(\mathrm{d}c) &= \int_0^\infty \left(\int_{\underline{\boldsymbol{p}}_D(c)}^\infty D(v)\,\mathrm{d}v\right) \boldsymbol{c}^{-1}(\mathrm{d}c) \\ &= \int_0^\infty \boldsymbol{c}^{-1}(\underline{\boldsymbol{p}}_D^{-1}(v))D(v)\,\mathrm{d}v \\ &= \int_0^\infty \boldsymbol{c}^{-1}(r_D(D(v)))D(v)\,\mathrm{d}v,\end{aligned}$$

as desired. ■

With Lemma 2, the results in Section 3 can now be used to derive topological properties of outcomes in this setting.

Corollary 1. *For any* $\boldsymbol{c} \in \mathcal{C}$, $\Pi^{\mathrm{II}}(\boldsymbol{c}, \cdot)$ *is continuous on* $\mathcal{D}$ *and* $\Sigma^{\mathrm{II}}(\boldsymbol{c}, \cdot)$ *is upper-semicontinuous on* $\mathcal{D}$. *Meanwhile, for any* $D \in \mathcal{D}$, $\Pi^{\mathrm{II}}(\cdot, D)$ *is continuous on* $\mathcal{C}$ *and* $\Sigma^{\mathrm{II}}(\cdot, D)$ *is upper-semicontinuous on* $\mathcal{C}$.

Proof. For any $\boldsymbol{c} \in \mathcal{C}$, continuity of $\Pi^{\text{II}}(\boldsymbol{c}, \cdot)$ and upper-semicontinuity of $\Sigma^{\text{II}}(\boldsymbol{c}, \cdot)$ follow from Proposition 4, Proposition 7, Lemma 2 and the dominated convergence theorem. For any $D \in \mathcal{D}$, continuity of $\Pi^{\text{II}}(\cdot, D)$ and upper-semicontinuity of $\Sigma^{\text{II}}(\cdot, D)$ follow from Proposition 4, Proposition 7, Lemma 2 and the Portmanteau theorem. ■

4.2 Third-Degree Price Discrimination

Consider a canonical third-degree price discrimination problem. The environment is the same as described in Section 3. However, the monopolist can further price discriminate the consumers through a given market segmentation. A market segmentation is a probability measure $s \in \Delta(\mathcal{D})$[5] such that for all $p \geq 0$,

$$\int_{\mathcal{D}} D(p)s(\mathrm{d}D) = D_0(p),$$

where $D_0 \in \mathcal{D}$ describes the market demand. Let $\mathcal{S}$ be the collection of market segmentations, and endow $\mathcal{S}$ with the weak-* topology.[6] The monopolist engages in third-degree price discrimination by charging different prices to different segment of consumers. Therefore, given marginal cost c and segmentation s, the monopolist's optimal profit is

$$\Pi^{\text{III}}(c, s) := \int_{\mathcal{D}} \pi_D(c)s(\mathrm{d}D),$$

whereas the consumer surplus is

$$\Sigma^{\text{III}}(c, s|\boldsymbol{p}) := \int_{\mathcal{D}} \left(\int_{\{v \geq \boldsymbol{p}_D(c)\}} (v - \boldsymbol{p}_D(c))^+ D(\mathrm{d}v) \right) s(\mathrm{d}D),$$

for any $\boldsymbol{p} \in \boldsymbol{P}$, and the induced output level is

$$Q^{\text{III}}(c, s|\boldsymbol{p}) := \int_{\mathcal{D}} D(\boldsymbol{p}_D(c))s(\mathrm{d}D),$$

for any $\boldsymbol{p} \in \boldsymbol{P}$

The continuity properties derived in Section 2 can be easily extended to this setting, as summarized by Corollary 2 below. The proof of Corollary 2, just as the proof of Corollary 1, is a direct application of the results in Section 2, the dominated convergence theorem, and the Portmanteau theorem and hence is omitted.

Corollary 2. *For any $s \in \mathcal{S}$, $\Pi^{\text{III}}(\cdot, s)$ is continuous, $\Sigma^{\text{III}}(\cdot, s|\overline{\boldsymbol{p}})$ and $Q^{\text{III}}(\cdot, s|\overline{p})$ are lower-semicontinuous; while $\Sigma^{\text{III}}(\cdot, s|\underline{\boldsymbol{p}})$ and $Q^{\text{III}}(\cdot, s|\underline{\boldsymbol{p}})$ are upper-semicontinuous. Furthermore, for any $c \geq 0$, $\Pi^{\text{III}}(c, \cdot)$ is continuous on $\mathcal{S}$, $\Sigma^{\text{III}}(c, \cdot|\overline{\boldsymbol{p}})$ and $Q^{\text{III}}(c, \cdot|\overline{p})$ are lower-semicontinuous on $\mathcal{S}$; while $\Sigma^{\text{III}}(c, \cdot|\underline{\boldsymbol{p}})$ and $Q^{\text{III}}(c, \cdot|\underline{\boldsymbol{p}})$ are upper-semicontinuous on $\mathcal{S}$.*

[5] $\mathcal{D}$ is endowed with the Borel σ-algebra generated by the weak-* topology

[6] Endow $\mathcal{D}$ the σ-algebra generated by the weak-* topology. The set of probability measures $\Delta(\mathcal{D})$ of $\mathcal{D}$ can then be endowed with the weak-* topology.

5 Conclusion

In this note, I outline various topological properties of outcomes in a monopolistic pricing setting, including the canonical monopolistic pricing problem, as well as environments that feature second and third degree price discrimination. These continuity results are methodologically relevant to economic models where the demand function and the cost structure are endogenous (including product design (Johnson and Myatt, 2006), hold-up problems (Condorelli and Szentes, 2020), information acquisition (Roesler and Szentes (2017), Ravid et al. (2019)) and the sale of consumer data (Yang, 2020)); or are objects in comparative statics analyses (Yang, 2019). Although this note focuses on monopolistic pricing problems with a single product, it is also worthwhile to explore topological properties of outcomes in many other environments, including competition, auction, and monopoly markets with multiple products. These are subjects for future research.

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