Online Appendix of
Market-Minded Informational Intermediary and Unintended Welfare Loss

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OA.1 Formal Definition of Histories and Strategies

For any period $t \geq 0$, the history of the seller at period t consists of all the prices charged by the seller until period $t-1$, all the disclosure policies adopted by the intermediary until period $t-1$, and all tie-breaking rules used by the tie-breaker until period $t-1$. Specifically, a history for the seller at period t is denoted by $\tilde{h}^t = \{p_s, D_s, q_s\}_{s=0}^{t-1}$, where $p_s \geq 0$ is the price charged by the seller in period s, $D_s \in \mathcal{D}$ is the disclosure policy adopted by the intermediary in period s, and $q_s \in [D_s(p_s^+), D_s(p_s)]$ is the tie-breaking rule adopted by the tie-breaker in period s. For any $t \geq 1$, let $\widetilde{\mathcal{H}}^t$ be the collection of sequences $\{p_s, D_s, q_s\}_{s=1}^{t-1}$ such that $p_s \geq 0$, $D_s \in \mathcal{D}$, and $q_s \in [D_s(p_s^+), D_s(p_s)]$ for all $s \in \{0, \dots, t-1\}$. Also, let $\widetilde{\mathcal{H}}^0 := \emptyset$. Finally, let $\widetilde{H} := \cup_{t=0}^{\infty} \widetilde{H}^t$.

In the meantime, in any period $t \geq 0$, the history of the intermediary is the history of the seller and the price charged by the seller in period t. That is, a history for the intermediary in period $t \geq 0$ denoted by $h^t = (\tilde{h}^t, p_t)$, for some $\tilde{h}^t \in \widetilde{H}^t$ and some $p_t \geq 0$. Let $\mathcal{H}^t$ be defined as $\widetilde{H}^t \times \mathbb{R}_+$ for all $t \geq 0$, and let $\mathcal{H} := \cup_{t=0}^{\infty} \mathcal{H}^t$.

Furthermore, in any period $t \geq 0$, the history of the tie-breaker is the history of the seller joint with the price charged by the seller and the disclosure policy adopted by the intermediary in period t. That is, a history for the tie-breaker in period $t \geq 0$ denoted by $\hat{h}^t = (h^t, D_t)$, for some $h^t \in H^t$ and some $D_t \geq \mathcal{D}$. Let $\widehat{\mathcal{H}}^t$ be defined as $H^t \times \mathcal{D}$ for all $t \geq 0$, and let $\widehat{\mathcal{H}} := \cup_{t=0}^{\infty} \widehat{\mathcal{H}}^t$.

With this definition, the seller's strategy can be formally defined as a measurable function from $\widetilde{H}$ to $\mathbb{R}_+$; the intermediary's strategy can be defined as a measurable function from $\mathcal{H}$ to $\mathcal{D}$; and the tie-breaker's strategy can be defined as a measurable function from $\widehat{H}$ to $[0, 1]$ such that the value of this function must be in $[D_t(p_t^+), D_t(p_t)]$ for any $(\tilde{h}^t, p_t, D_t) \in \widehat{\mathcal{H}}^t$, for all t. Finally, for any histories $h^t, h^s \in \mathcal{H}$ for the intermediary, we say that h^t is a predecessor of h^s if $s > t$ and $h^s = (h^t, \{D_\tau, q_\tau, p_{\tau+1}\}_{\tau=t}^{s-1})$, and we say that $\sigma|_{h^t}$ is a continuation strategy of the intermediary if there exists a strategy σ of the intermediary such that $\sigma(h^s) = \sigma|_{h^t}(h^s)$ if h^t is a predecessor of h^s.

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OA.2 Omitted Proofs for Section 5

OA.2.1 Proof of Lemma 5

Consider any subgame perfect equilibrium. At any history h^t, choosing a myopically optimal demand in every future periods is always a feasible strategy. Suppose that the intermediary deviates to this strategy. Notice that although the seller's prices may change after this deviation even if the seller's strategy remains unchanged (as the history of the play may be different), the seller must always be best responding in each period as he is short-lived. Therefore, prices in each period must be within $[r^*, p^*]$. As a result, the intermediary's stage game payoff after this deviation must be at least

$$\min_{p\in[r^*,p^*]} \alpha p\overline{D}(v^{-1}(p)) = \alpha p^*\overline{D}(v^{-1}(p^*)) = r^*.$$

Hence, the intermediary's continuation payoff at history h^t in this equilibrium must be at least

$$\frac{\alpha r^*}{1-\gamma\delta} = \omega^*.$$

If $\underline{\omega} = \omega^*$, then the proof is complete. Thus, it is without loss to assume that $\underline{\omega} > \omega^*$. Now consider any subgame perfect equilibrium and history h^t. Let $\omega_0 := \omega^*$. For any $n \in \mathbb{N}$, let

$$\omega_n := h(\omega_{n-1}) = \delta\left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\omega_{n-1}}\right)p^*}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\omega_{n-1}.$$

Since $\omega_0 = \omega^* < \underline{\omega}$, it must be that $\omega_n > h(\omega_n)$ for all $n \in \mathbb{N}$. Moreover, since h is increasing, $h(\omega) \leq h(\underline{\omega})$ for all $\omega \leq \underline{\omega}$. Hence, since $\omega^* < \underline{\omega}$, $\{\omega_n\}$ is an increasing sequence and is bounded from above by $\underline{\omega}$. Therefore, $\lim_{n\to\infty}\omega_n$ exists and is finite. Furthermore, since h is continuous,

$$\lim_{n\to\infty}\omega_n = \lim_{n\to\infty}h(\omega_{n-1}) = h(\lim_{n\to\infty}\omega_{n-1}) = h(\lim_{n\to\infty}\omega_n).$$

As $\underline{\omega} > \omega^*$ and hence $\underline{\omega}$ is the unique solution of $\omega = h(\omega)$, it must be that $\lim_{n\to\infty}\omega_n = \underline{\omega}$. We claim that the intermediary's continuation payoff at any history in any subgame perfect equilibrium must be at least ω_n for all $n \in \mathbb{N}\cup\{0\}$, which will in turn imply that her equilibrium payoff is at least $\underline{\omega}$.

We prove this claim by induction. Consider any subgame perfect equilibrium, since $\omega_0 = \omega^*$, the intermediary's continuation payoff at any history must be at least ω_0. For any $n \in \mathbb{N}$, suppose that the intermediary's continuation payoff at any history is at least ω_{n-1}. It suffices to show that her continuation payoff at any history is at least ω_n as well. To this end, consider any history h^t at any period t. Suppose that the seller charges p. Then, since the intermediary's continuation payoff starting from the next period is at least ω_{n-1}, regardless of the outcomes in this period, her continuation payoff at (h^t, p) must be at least

$$\mathbf{W}(p|\omega_{t-1}) = \sup_{D\in\mathcal{D}}\left[\alpha p D(p) + \delta\left(\gamma + \beta\int_p^{\infty} D(v)\,\mathrm{d}v\right)\omega_{n-1}\right].$$

Since price p must be the seller's best response given the intermediary's strategy, it must be that $p \leq p^*$. Moreover, since $\mathbf{W}(p|\omega) = \alpha p\overline{D}(v^{-1}(p))$ whenever $\alpha \geq \delta\beta\omega_{n-1}$ and $\mathbf{W}(p|\omega)$ is decreasing on $[0, p^*]$ whenever

$\alpha < \delta\beta\omega_{n-1}$ (see Lemma OA.1 below), it must be that $\mathbf{W}(p|\omega_{n-1}) \geq \mathbf{W}(p^*|\omega_{n-1})$ for all $p \leq p^*$. Therefore, at any history h^t, the seller's continuation payoff must be at least

$$\mathbf{W}(p^*|\omega_{n-1}) = \sup_{D\in\mathcal{D}} \left[\alpha p^* D(p^*) + \delta\left(\gamma + \beta\int_{p^*}^{\infty} D(v)\,\mathrm{d}v\right)\omega_{n-1}\right].$$

Moreover, by Lemma 3,

$$\begin{aligned}
\mathbf{W}(p^*|\omega_{t-1}) =& \alpha p^*\overline{D}(\xi(p^*|\omega_{n-1})) \\
&+ \delta\left(\gamma + \beta\left(\int_{\xi(p^*|\omega_{n-1})}^{\infty} \overline{D}(v)\,\mathrm{d}v - (p - \xi(p^*|\omega_{n-1}))p^*\overline{D}(\xi(p^*|\omega_{n-1}))\right)\right)\omega_{n-1} \\
=& \max\left\{\alpha p^*\overline{D}(v^{-1}(p^*)) + \delta\left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\omega_{n-1}}\right)p^*}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\omega_{n-1}\right\} \\
=& \max\{\alpha p^*\overline{D}(v^{-1}(p^*)) + \gamma\delta\omega_{n-1}, h(\omega_{n-1})\} \\
=& h(\omega_{n-1}) \\
=& \omega_n,
\end{aligned}$$

where the second to the last equality follows from the facts that $\{\omega_n\}$ is an increasing sequence and that $h(\omega_n) > \omega_n$ for all n, which in turn imply that

$$\alpha p^*\overline{D}(v^{-1}(p^*)) + \gamma\delta\omega_{n-1} = (1-\gamma\delta)\omega^* + \gamma\delta\omega_{n-1} = (1-\gamma\delta)\omega_0 + \gamma\delta\omega_{n-1} \leq \omega_{n-1} < h(\omega_{n-1}).$$

This completes the proof. ■

OA.2.2 Proof of Theorem 3

The proof of Theorem 3 involves some additional definitions and lemmas as follows.

Lemma OA.1. *Suppose that $\beta < \beta^*$. Then for any $\omega \geq 0$, $\mathbf{W}(p|\omega) = \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\omega$ for all $p \geq 0$ whenever $\alpha \geq \delta\beta\omega$. Meanwhile, if $\alpha < \delta\beta\omega$, then $\mathbf{W}(\cdot|\omega)$ is differentiable except at countably many $p \in [0, p^*]$, is decreasing on $[0, p^*]$, and $\mathbf{W}(p^*|\underline{\omega}) = \underline{\omega}$.*

Proof. By Lemma 3, for any $p \geq 0$

$$\begin{aligned}
\mathbf{W}(p|\omega) &= \max_{\xi\in[v^{-1}(p),p]} \left[\alpha p\overline{D}(\xi) + \delta\left(\gamma + \beta\left(\int_{\xi}^{\infty}\overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi)\right)\omega\right)\right] \\
&= \left[\alpha p\overline{D}(\xi(p|\omega)) + \delta\left(\gamma + \beta\left(\int_{\xi(p|\omega)}^{\infty}\overline{D}(v)\,\mathrm{d}v - (p-\xi(p|\omega))\overline{D}(\xi(p|\omega))\right)\right)\omega\right].
\end{aligned}$$

Therefore, $\mathbf{W}(p|\omega) = \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\omega$ if $\alpha \geq \delta\beta\omega$.

In the meantime, if $\alpha < \delta\beta\omega$, then $\mathbf{W}(\cdot|\omega)$ is continuous on $[0, p^*]$, and is differentiable at all $p \in [0, p^*]$ except for those such that $(1-\alpha/\delta\beta\omega)^+ p = v^{-1}(p)$, which is at most a countable subset of $[0, p^*]$. Therefore, by Lemma 1 of Milgrom and Segal (2002),

$$\frac{\partial}{\partial p}\mathbf{W}(p|\omega) = (\alpha - \delta\beta\omega)\overline{D}(\xi(p|\alpha,\beta,\delta,\omega)) \leq 0,$$

for all $p \in [0, p^*]$ at which $(1 - \alpha/\delta\beta\omega)p \neq v^{-1}(p)$ and hence $\mathbf{W}(\cdot|\omega)$ is decreasing on $[0, p^*]$. Now notice that if $\underline{\omega} = \omega^*$, then

$$\mathbf{W}(p^*|\underline{\omega}) = \sup_{D \in \mathcal{D}} \left[\alpha p^* D(p^*) + \delta \left(\gamma + \beta \int_{p^*}^{\infty} D(v)\, \mathrm{d}v \right) \underline{\omega} \right] \geq \alpha p^* \overline{D}(v^{-1}(p^*)) + \gamma\delta\underline{\omega} = \underline{\omega}$$

Meanwhile, if $\omega^* > \underline{\omega}$, Lemma 3 then implies that

$$\mathbf{W}(p|\underline{\omega}) = \delta \left(\gamma + \beta \int_{\left(1 - \frac{\alpha}{\delta\beta\underline{\omega}}\right)p^*}^{\infty} \overline{D}(v)\, \mathrm{d}v \right) \underline{\omega} > \underline{\omega},$$

and hence $h(\underline{\omega}) > 1$, which contradicts to $\beta < \beta^*$. Meanwhile, if $\underline{\omega} > \omega^*$, then since

$$\alpha p^* \overline{D}(v^{-1}(p^*)) + \gamma\delta\underline{\omega} = (1 - \gamma\delta)\omega^* + \gamma\delta\underline{\omega} < \underline{\omega}$$

and since $h(\underline{\omega}) = \underline{\omega}$, by Lemma 3, we must have

$$\mathbf{W}(p^*|\underline{\omega}) = \delta \left(\gamma + \beta \int_{\left(1 - \frac{\alpha}{\delta\beta\underline{\omega}}\right)p^*}^{\infty} \overline{D}(v)\, \mathrm{d}v \right) \underline{\omega} = h(\underline{\omega}) = \underline{\omega}.$$

This completes the proof. ■

Lemma OA.2. *Suppose that $\beta \leq \underline{\beta}$. Then in any finite subgame perfect equilibrium, the intermediary's equilibrium payoff is at most $\alpha\mathbb{E}[v]/(1 - \gamma\delta)$.*

Proof. Consider any finite subgame perfect equilibrium. Let ω denote the intermediary's equilibrium payoff. According to the structure of the game, ω can be written as

$$\omega = \alpha p_0 D_0(p_0) + \sum_{t=1}^{\infty} \delta^t \prod_{s=0}^{t-1} \left(\gamma + \beta \int_{p_s}^{\infty} D_s(v)\, \mathrm{d}v \right) \alpha p_t D_t(p_t),$$

for some sequence $\{p_t, D_t\}$ with $p_t \geq 0$ and $D_t \in \mathcal{D}$ for all t. In the meantime, for any $T \in \mathbb{N}$, let

$$\omega_T := \sup_{\{\tilde{D}_t, \tilde{p}_t\}_{t=0}^{\infty}} \left[\alpha \tilde{p}_0 \tilde{D}_0(p_0) + \sum_{t=1}^{T} \delta^t \prod_{s=0}^{t-1} \left(\gamma + \beta \int_{\tilde{p}_s}^{\infty} \tilde{D}_s(v)\, \mathrm{d}v \right) \alpha \tilde{p}_t \tilde{D}_t(\tilde{p}_t) \right]$$

For any $T \in \mathbb{N}$, note ω_T can be recursively written as

$$\omega_T = \sup_{\tilde{p} \geq 0, \tilde{D} \in \mathcal{D}} \left[\alpha \tilde{p} \tilde{D}(p) + \delta \left(\gamma + \beta \int_{\tilde{p}}^{\infty} \tilde{D}(v)\, \mathrm{d}v \right) \omega_{T-1} \right]$$

where $\omega_0 := \alpha\mathbb{E}[v]$.

As a result, since $\omega < \infty$, and since the seller's payoff in each period must be at least r^* in any finite subgame perfect equilibrium, it must be that

$$\omega \leq \limsup_{T \to \infty} \omega_T.$$

Now fix any $T \in \mathbb{N}$ and suppose that $\omega_{T-\tau} = \alpha \sum_{s=0}^{\tau} \delta^s \mathbb{E}[v]$ for some $\tau < T$. Then since $\beta \leq \underline{\beta}$,

$$\delta\beta\omega_{T-\tau} \leq \delta\beta \frac{\alpha\mathbb{E}[v]}{1 - \gamma\delta} \leq \alpha.$$

Therefore, for any $p \geq 0$, $\xi(p|\omega^*_{T-\tau}) = v^{-1}(p)$. Hence, by Lemma 3,

$$\begin{aligned}\omega_{T-\tau-1} &= \sup_{p\in[r^*,p^*]} [\alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\omega_{T-\tau}] \\ &= \sum_{s=0}^{\tau+1} \gamma^s\delta^s\alpha\mathbb{E}[v].\end{aligned}$$

Therefore, by induction, it must be that

$$\omega_{T-\tau} = \sum_{s=0}^{\tau} \gamma^s\delta^s\alpha\mathbb{E}[v],$$

for all $\tau \in \{0,\ldots,T\}$. In particular,

$$\omega_T = \sum_{s=0}^{T} \delta^s\alpha\mathbb{E}[v].$$

Therefore,

$$\omega \leq \limsup_{T\to\infty} \omega_T = \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta},$$

as desired. ■

Lemma OA.3. *Suppose that $\beta \in (\underline{\beta}, \widehat{\beta})$, and that in any finite subgame perfect equilibrium, the total revenue in each period is at least $\underline{r}$. Then in any finite subgame perfect equilibrium, the intermediary's normalized continuation value at any history is at most*

$$u(\underline{r}) := \frac{\alpha}{\delta\beta}\frac{\underline{r}}{\underline{r} - p^\beta\overline{D}(p^\beta)}. \tag{OA.1}$$

Proof. Consider any subgame perfect equilibrium and any history. Let ω denote the intermediary's normalized continuation payoff at this history. According to the structure of the game, ω can be written as

$$\omega = \alpha p_0 D_0(p_0) \sum_{t=1}^{\infty} \delta^t \prod_{s=0}^{t-1} \left(\gamma + \beta \int_{p_s}^{\infty} D_s(v)\,\mathrm{d}v\right) \alpha p_t D_t(p_t),$$

for some sequence $\{p_t, D_t\}$ with $p_t \geq 0$ and $D_t \in \mathcal{D}$ for all t. In the meantime, for any $T \in \mathbb{N}$, let

$$\begin{aligned}\omega_T := \sup_{\{\tilde{D}_t,\tilde{p}_t\}_{t=0}^{\infty}} & \left[\alpha\tilde{p}_0\tilde{D}_0(\tilde{p}_0) + \sum_{t=1}^{T} \delta^t \prod_{s=0}^{t-1} \left(\gamma + \beta \int_{\tilde{p}_s}^{\infty} \tilde{D}_s(v)\,\mathrm{d}v\right) \alpha\tilde{p}_t\tilde{D}_t(p_t)\right] \\ \text{s.t. } & \tilde{p}_t\tilde{D}_t(\tilde{p}_t) \geq \underline{r}, \text{ for all } t.\end{aligned}$$

Moreover, notice that for any $T \in \mathbb{N}$, ω_T can be recursively written as

$$\begin{aligned}\omega_T = \sup_{\tilde{p}\geq 0,\tilde{D}\in\mathcal{D}} & \left[\alpha\tilde{p}\tilde{D}(\tilde{p}) + \delta\left(\gamma + \beta\int_{\tilde{p}}^{\infty} \tilde{D}(v)\,\mathrm{d}v\right)\omega_{T-1}\right] \\ \text{s.t. } & \tilde{p}\tilde{D}(\tilde{p}) \geq \underline{r}\end{aligned}$$

where $\omega_0 := \alpha\mathbb{E}[v]$.

Lastly, since $\omega < \infty$, and since the total revenue in each period must be at least $\underline{r}$ in any subgame perfect equilibrium, it must be that

$$\omega \leq \limsup_{T\to\infty} \omega_T.$$

Since $\beta > \underline{\beta}$, $\alpha\mathbb{E}[v]/(1-\gamma\delta) > \alpha/\delta\beta$, and hence there exists $\bar{\tau}$ such that $\sum_{s=0}^{\tau} \gamma^s\delta^s\alpha\mathbb{E}[v] > \alpha/\delta\beta$ for all $\tau \geq \bar{\tau}$. Now consider any $T > \bar{\tau}$. For any $\tau \in \{1, \dots, \bar{\tau}\}$, any $p \geq 0$, and for any $\omega_{T-\tau} > \alpha$, by Lemma 3,

$$\begin{aligned}\omega_{T-\tau+1} = \max_{p\geq 0} \Bigg[&\alpha p\overline{D}(\xi(p|\omega_{T-\tau})) \\ &+ \delta\left(\gamma + \beta\left(\int_{\xi(p|\omega_{T-\tau})}^{\infty} \overline{D}(v)\,\mathrm{d}v - (p - \xi(p|\omega_{T-\tau}))\overline{D}(\xi(p|\omega_{T-\tau}))\right)\right)\omega_{T-\tau}\Bigg] \\ &\text{s.t. } p\overline{D}(\xi(p|\omega_{T-\tau})) \geq \underline{r},\end{aligned}$$

where

$$\begin{aligned}\xi(p|\omega_{T-\tau}) &= \max\left\{\left(1 - \frac{\alpha}{\delta\beta\omega_{T-\tau}}\right)p, v^{-1}(p)\right\} \\ &= \begin{cases} \left(1 - \frac{\alpha}{\delta\beta\omega_{T-\tau}}\right)p, & \text{if } \omega_{T-\tau} \geq g^{\beta}\left(\left(1 - \frac{\alpha}{\delta\beta\omega_{T-\tau}}\right)p\right) \\ v^{-1}(p), & \text{if } \omega^*_{T-\tau} \leq g^{\beta}\left(\left(1 - \frac{\alpha}{\delta\beta\omega^*_{T-\tau}}\right)p\right) \end{cases},\end{aligned}$$

for all $p \geq 0$, which in turn can be written as

$$\begin{aligned}\omega_{T-\tau+1} = \max_{p\in[0,\overline{p}]} &\ \psi(p|\omega_{T-\tau}) \\ \text{s.t. } &p\overline{D}(\xi(p|\omega_{T-\tau})) \geq \underline{r}.\end{aligned}$$

where

$$\psi(p|\tilde{\omega}) := \begin{cases} \delta\left(\gamma + \beta\int_p^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\tilde{\omega}, & \text{if } g^{\beta}(p) \geq \tilde{\omega} \\ \frac{\alpha\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega}-\alpha}p\overline{D}\left(v^{-1}\left(\frac{\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega}-\alpha}p\right)\right) + \gamma\delta\tilde{\omega}, & \text{if } g^{\beta}(p) < \tilde{\omega} \end{cases},$$

for all $p \geq 0$ and for all $\tilde{\omega} \geq 0$. Therefore, suppose that for some $\tau \in \{1, \dots, \bar{\tau}\}$, $\omega_{T-\tau} > \alpha/\delta\beta$. Then

$$\omega_{T-\tau+1} \geq \alpha\mathbb{E}[v] + \gamma\delta\omega_{T-\tau} > \frac{\alpha}{\delta\beta}.$$

By induction, it then follows that $\omega_{T-\tau} > \alpha/\delta\beta$ for all $\tau \in \{1, \dots, \bar{\tau}\}$.

Furthermore, since g^{β} is increasing on $[0, \overline{p}]$, the functions

$$p \mapsto \delta\left(\gamma + \beta\int_p^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\tilde{\omega}$$

and

$$p \mapsto \frac{\alpha\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega} - \alpha}p\overline{D}\left(v^{-1}\left(\frac{\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega} - \alpha}p\right)\right)$$

cross at most once in $[0, \overline{p}]$ for any $\tilde{\omega} > \alpha/\delta\beta$, and whenever they cross, it must be that

$$p \mapsto \delta\left(\gamma + \beta\int_p^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\tilde{\omega}$$

is decreasing at the crossing point. Therefore, if these two functions cross and the crossing point is below p^β, then it must be that $\omega_{T-\tau+1} = \alpha\mathbb{E}[v] + \gamma\delta\omega_{T-\tau} > \omega_{T-\tau}$. Otherwise, it must be that

$$\omega_{T-\tau+1} = \delta\left(\gamma + \beta\int_{p_{T-\tau}}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\omega_{T-\tau},$$

where $p_{T-\tau} \in [0, \hat{p}]$ is the unique solution of

$$\frac{\delta\beta\omega_{T-\tau}}{\delta\beta\omega_{T-\tau} - \alpha} p\overline{D}(p) = \underline{r}.$$

Meanwhile, notice that since $\alpha\mathbb{E}[v]/(1-\gamma\delta) > \alpha/\delta\beta$, the crossing point must be above p^β for T large enough and τ small enough.

Finally, for any $\tilde{\omega} > \alpha/\delta\beta$, let $p(\tilde{\omega})$ be the unique solution of

$$\frac{\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega} - \alpha} p\overline{D}(p) = \underline{r}.$$

and let

$$\Pi(\tilde{\omega}) := \tilde{\omega}\left(\gamma + \beta\int_{p(\tilde{\omega})}^{\infty} \overline{D}(v)\,\mathrm{d}v\right).$$

Notice that by definition, $\Pi(\tilde{\omega}) > \tilde{\omega}$ if $\tilde{\omega} < u(\underline{r})$; $\Pi(\tilde{\omega}) < \tilde{\omega}$ if $\tilde{\omega} > u(\underline{r})$ and $\Pi(u(\underline{r})) = u(\underline{r})$. Together, it then follows that for any T large enough,

$$\omega_T = \Pi(\omega_{T-1}).$$

Therefore, $\{\omega_T\}$ has a limit and $\lim_{T\to\infty} \omega_T = u(\underline{r})$, and hence

$$\omega \leq \lim_{T\to\infty} \omega_T = u(\underline{r}),$$

as desired. ■

Proof of Theorem 3. We first introduce a family of regimes that describe the intermediary's and the seller's strategies. Then we will use these regimes to construct subgame perfect equilibria to support the desired payoffs,

> **Regime** p-MYOPIC: The seller plays p. The intermediary chooses any $D \in \Delta(p|0)$. The tie-breaker then breaks ties in favor of the seller (i.e. the tie breaker chooses $q = D(p)$).
>
> **Regime** (p, ω)-TRANSITION: The seller plays p, the intermediary chooses any $D \in \Delta(p|\omega)$ and the tie-breaker breaks ties in favor of the seller.
>
> **Regime** D-PUNISH: The intermediary chooses D after seeing the seller's price, and the tie-breaker breaks ties against the seller.

We now characterize the set of equilibrium payoffs by four cases separately.

Case 1: $\beta \leq \underline{\beta}$.

In this case, since

$$\frac{1-\gamma\delta}{\delta\beta} \geq \mathbb{E}[v] > p^*\overline{D}(v^{-1}(p^*)),$$

$\delta\beta\omega^* > \alpha$. Thus,

$$h(\omega^*) = \delta(1+\beta\mathbb{E}[v])\omega^* \leq \omega^*,$$

which in turn implies, by concavity of h, $\underline{\omega} = \omega^* < \alpha/\delta\beta$.

Now let $\overline{\omega} := \alpha\mathbb{E}[v]/(1-\gamma\delta)$. For any $p \geq 0$ and for any $\tilde{\omega} \in [\underline{\omega}, \overline{\omega}]$, let $\Lambda^\beta(p,\tilde{\omega})$ be the value of the following constraint maximization problem:

$$\sup_{D\in\mathcal{D}} \left[\alpha pD(p) + \delta\left(\gamma + \beta\int_p^\infty D(v)\,\mathrm{d}v\right)\overline{\omega}\right] \tag{OA.2}$$

$$\text{s.t. } \frac{\alpha pD(p)}{1-\gamma\delta} \leq \tilde{\omega}. \tag{OA.3}$$

Note that if the constraint does not bind,

$$\Lambda^\beta(p,\tilde{\omega}) = \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\overline{\omega},$$

while if the constraint binds,

$$\Lambda^\beta(p,\tilde{\omega}) = \lambda(1-\gamma\delta)\tilde{\omega} + \delta\left(\gamma + \beta\int_{\left(1-\frac{(1-\lambda)\alpha}{\delta\beta\overline{\omega}}\right)p}^\infty \overline{D}(v)\,\mathrm{d}v\right)\overline{\omega},$$

and

$$\frac{\partial}{\partial p}\Lambda^\beta(p,\underline{\omega}) = ((1-\lambda)\alpha - \delta\beta\overline{\omega})\overline{D}\left(\left(1-\frac{(1-\lambda)\alpha}{\delta\beta\overline{\omega}}\right)p\right) \leq 0,$$

by the envelope theorem, where λ is the Lagrangian multiplier that solves

$$\frac{\alpha p\overline{D}\left(\left(1-\frac{(1-\lambda)\alpha}{\delta\beta\overline{\omega}}\right)\right)}{1-\gamma\delta} = \tilde{\omega}.$$

Since $\tilde{\omega} \in [\underline{\omega}, \overline{\omega}]$, there exists $\tilde{p} \in [\overline{p}, p^*]$ such that

$$\frac{\alpha\overline{D}(v^{-1}(\tilde{p}))}{1-\gamma\delta} = \tilde{\omega}.$$

This implies that the constraint of (OA.2) binds if and only if $p \geq \tilde{p}$. Together, $\Lambda^\beta(\cdot,\tilde{\omega})$ is a decreasing function on $[0,p^*]$.

In the meantime, for any $\tilde{\omega} \in [\underline{\omega}, \overline{\omega}]$, notice that the function

$$p \mapsto \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega}$$

is quasi-concave and has a maximum at $p = \mathbb{E}[v]$. Moreover, since for any $\lambda \in [1-\delta\beta\overline{\omega}/\alpha, 1]$ and for any $p \in [0,p^*]$,

$$\delta\beta\overline{\omega}\int_{\left(1-\frac{\alpha(1-\lambda)}{\delta\beta\overline{\omega}}\right)p}^\infty \overline{D}(v)\,\mathrm{d}v \geq (1-\lambda)\alpha p\overline{D}(v^{-1}(p))$$

if and only if

$$\overline{\omega} \geq g\left(\left(1 - \frac{\alpha(1-\lambda)}{\delta\beta\overline{\omega}}\right)p\right),$$

the functions

$$p \mapsto \delta\beta\overline{\omega}\int_{\left(1-\frac{\alpha(1-\lambda)}{\delta\beta\overline{\omega}}\right)p}^{\infty} \overline{D}(v)\,\mathrm{d}v$$

and

$$p \mapsto \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega}$$

cross at most once.

Together, for any $\tilde{\omega} \in [\underline{\omega}, \overline{\omega}]$,

$$\Lambda^{\beta}(p, \tilde{\omega}) \geq \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega},\ \forall p \geq 0$$

if and only if

$$\Lambda^{\beta}(\mathbb{E}[v], \tilde{\omega}) \geq \alpha\mathbb{E}[v] + \gamma\delta\tilde{\omega}.$$

We now define $\underline{\boldsymbol{\omega}}^f(\beta)$ as

$$\underline{\boldsymbol{\omega}}^f(\beta) := \inf\{\tilde{\omega} \in [\underline{\omega}, \overline{\omega}] | \Lambda^{\beta}(\mathbb{E}[v], \tilde{\omega}) \geq \alpha\mathbb{E}[v] + \gamma\delta\tilde{\omega}\}.$$

Notice that $\underline{\boldsymbol{\omega}}^f(\beta)$ is well-defined since

$$\Lambda^{\beta}(\mathbb{E}[v], \overline{\omega}) \geq \alpha\mathbb{E}[v] + \gamma\delta\overline{\omega}.$$

Moreover, by definition, since Λ^{β} is increasing in β, $\underline{\boldsymbol{\omega}}^f$ is nonincreasing in β on $[0, \underline{\beta}]$.

We now claim that in any subgame perfect equilibrium, the total revenue in each period must be at least $\underline{r} := (1-\gamma\delta)\underline{\boldsymbol{\omega}}^f(\beta)/\alpha$. Indeed, suppose the contrary, that there exists a subgame perfect equilibrium in which the lowest total revenue on the equilibrium path, say $\tilde{r}$, is strictly below $\underline{r}$. Then, since the seller must be best responding in the period where the total revenue is $\tilde{r}$, for any $p \geq 0$, there must exist $D \in \mathcal{D}$ such that

$$pD(p) \leq \tilde{r},$$

so that the intermediary would respond by choosing $D \in \mathcal{D}$ if the seller deviates to any price $p \geq 0$. Let $\omega(p, D)$ denote the continuation value at this history. In the meantime, at any such history, since it is always feasible for the intermediary to choose the myopically optimal demand, and since her continuation payoff must be at least α share of the present discounted value of the sum of total revenues onward, which in turn, by hypothesis, is no less than $\tilde{r}$, it must be that

$$\alpha pD(p) + \delta\left(\gamma + \beta\int_p^{\infty} D(v)\,\mathrm{d}v\right)\overline{\omega} \geq \alpha pD(p) + \delta\left(\gamma + \beta\int_p^{\infty} D(v)\,\mathrm{d}v\right)\omega(p, D) \geq \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\frac{\alpha\tilde{r}}{1-\gamma\delta},$$

where the first inequality follows from Lemma OA.2. Therefore, let $\tilde{\omega} := \alpha\tilde{r}/(1-\gamma\delta)$, it follows that for any $p \geq 0$, there exists D such that

$$\alpha pD(p) + \delta\left(\gamma + \beta\int_p^{\infty} D(v)\,\mathrm{d}v\right)\overline{\omega} \geq \alpha p\overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega}$$

and that

$$\frac{\alpha p D(p)}{1-\gamma\delta} \leq \tilde{\omega}$$

and hence

$$\Lambda^{\beta}(p,\tilde{\omega}) \geq \alpha p \overline{D}(v^{-1}(p)) + \gamma\delta\tilde{\omega}$$

for all $p \geq 0$, but $\tilde{\omega} < \underline{\boldsymbol{\omega}}^f(\beta)$, which contradicts to the definition of $\underline{\boldsymbol{\omega}}^f(\beta)$. Thus, in any subgame perfect equilibrium, the total revenue in each period must be at least $\underline{r} := (1-\gamma\delta)\underline{\boldsymbol{\omega}}^f(\beta)/\alpha$. This in turn implies that the intermediary's continuation payoff at any history must be at least

$$\frac{\alpha \underline{r}}{1-\gamma\delta} = \underline{\boldsymbol{\omega}}^f(\beta).$$

Together with Lemma OA.2, by letting $\overline{\boldsymbol{\omega}}^f(\beta) := \overline{\omega} = \alpha\mathbb{E}[v]/(1-\gamma\delta)$, it then follows that the intermediary's equilibrium payoff must be at least $\underline{\boldsymbol{\omega}}^f(\beta)$ and at most $\overline{\boldsymbol{\omega}}^f(\beta)$.

Now consider any $\hat{\omega} \in [\underline{\boldsymbol{\omega}}^f(\beta), \overline{\boldsymbol{\omega}}^f(\beta)]$. By continuity of the function $p \mapsto p\overline{D}(v^{-1}(p))$, there exists $\hat{p}, \underline{p}$ such that $\mathbb{E}[v] \leq \hat{p} \leq \underline{p} \leq p^*$ and that $\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))/(1-\gamma\delta) = \hat{\omega}$, $\alpha\underline{p}\overline{D}(v^{-1}(\underline{p}))/(1-\gamma\delta) = \underline{\boldsymbol{\omega}}^f(\beta)$. Moreover, for any $p \geq 0$, fix any solution of (OA.2) with $\tilde{\omega} = \underline{\boldsymbol{\omega}}^f(\beta)$ and denote it by D_p. We claim that the following strategy profile constitutes a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$:

- Start by playing regime $\hat{p}$-MYOPIC. If the seller deviates to $p' \neq \hat{p}$, then enter regime $D_{p'}$-PUNISH immediately. If the intermediary deviates, then move to regime $\underline{p}$-MYOPIC. Otherwise, stay in the same regime.

- For any $p' \geq 0$, under regime $D_{p'}$-PUNISH, if the intermediary deviates, then move to regime $\underline{p}$-MYOPIC. Otherwise, move to regime $\mathbb{E}[v]$-MYOPIC.

- Under regime $\underline{p}$-MYOPIC and regime $\mathbb{E}[v]$-MYOPIC, if the seller deviates to $p' \neq p$, then enter regime $D_{p'}$-PUNISH immediately. Otherwise, stay in the same regime.

To see that this is a subgame perfect equilibrium, first notice that the intermediary's continuation payoff induced by the strategy profile is finite in every subgame. Therefore, Lemma 1 implies that it suffices to show there are no incentives to deviate in each regime. Indeed, for any $p \in [\mathbb{E}[v], \underline{p}]$, under regime p-MYOPIC, if all players follow their strategies, the seller's revenue must be $(1-\alpha)p\overline{D}(v^{-1}(p))$. Meanwhile, given that the intermediary follows her strategy, if the seller deviates to $p' \neq p$, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is a solution of (OA.2) with $\tilde{\omega} = \underline{\boldsymbol{\omega}}^f(\beta)$, it must be that

$$(1-\alpha)p'D_{p'}(p') \leq \frac{(1-\alpha)(1-\gamma\delta)}{\alpha}\underline{\boldsymbol{\omega}}^f(\beta) = (1-\alpha)\underline{p}\overline{D}(v^{-1}(\underline{p})) \leq (1-\alpha)p\overline{D}(v^{-1}(p)).$$

Therefore, the seller does not have any incentive to deviate. As for the intermediary, in each period, the present discounted value of playing according to regime $\hat{p}$-MYOPIC is $\hat{\omega} = \alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))/(1-\gamma\delta)$, while the present discounted value of the best payoff she can obtain by deviating is

$$\sup_{D\in\mathcal{D}}\left[\alpha\hat{p}D(\hat{p}) + \delta\left(\gamma + \beta\int_{\hat{p}}^{\infty} D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^f(\beta)\right],$$

which, by Lemma 3 and by the fact that $\underline{\boldsymbol{\omega}}^f(\beta) \leq \alpha\mathbb{E}[v]/(1-\gamma\delta) < \alpha/\delta\beta$, is given by

$$\alpha\hat{p}\overline{D}(v^{-1}(\hat{p})) + \gamma\delta\underline{\boldsymbol{\omega}}^f(\beta) \leq \alpha\hat{p}\overline{D}(v^{-1}(\hat{p})) + \frac{\gamma\delta\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))}{1-\gamma\delta} = \hat{\omega},$$

where the last equality follows from $\underline{\boldsymbol{\omega}}^f(\beta) = \alpha \underline{p}\overline{D}(v^{-1}(\underline{p}))/(1-\gamma\delta) \leq \alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))/(1-\gamma\delta)$. Thus, the intermediary does not have an incentive to deviate either.

In the meantime, for any $p' \geq 0$, under regime $D_{p'}$-PUNISH, if the intermediary follows the strategy, her continuation payoff would be $\overline{\omega}$, whereas if she deviates, her payoff would be at most $\alpha p'\overline{D}(v^{-1}(p'))+\gamma\delta\underline{\boldsymbol{\omega}}^f(\beta)$. Since $D_{p'}$ is a solution of (OA.2) with $\tilde{\omega} = \underline{\boldsymbol{\omega}}^f(\beta)$, it must be that

$$\alpha p' D_{p'}(p') + \delta\left(\gamma + \beta\int_{p'}^{\infty} D_{p'}(v)\,\mathrm{d}v\right)\overline{\omega} = \Lambda^{\beta}(p', \underline{\boldsymbol{\omega}}^f(\beta)) \geq \alpha p'\overline{D}(v^{-1}(p')) + \delta\underline{\boldsymbol{\omega}}^f(\beta),$$

where the last inequality follows from the definition of $\underline{\boldsymbol{\omega}}^f(\beta)$. Thus, the intermediary does not have any incentives to deviate under this regime.

Together, the strategy profile described above is indeed a subgame perfect equilibrium. Moreover, the intermediary's payoff in this equilibrium is $\hat{\omega}$.

Case 2: $\delta\beta/(1-\gamma\delta) \in (\underline{\beta}, \widehat{\beta})$.

We first claim that $\underline{\omega} = \omega^*$ in this case as well. To see this, recall that since $\beta \in (\underline{\beta}, \overline{\beta}]$, there exists $p^{\beta} \in (0, \overline{p}]$ such that

$$\int_{p^{\beta}}^{\infty} \overline{D}(v)\,\mathrm{d}v = \frac{1-\gamma\delta}{\delta\beta}. \tag{OA.4}$$

Let $\omega^{\beta} := g^{\beta}(p^{\beta})$. Lemma 4 then implies that

$$\frac{\delta\beta\omega^{\beta}}{\delta\beta\omega^{\beta}-\alpha}p^{\beta} \in \operatorname*{argmax}_{p\geq 0} p\mathbf{D}(p|p),$$

for any selection $\mathbf{D}$ of $\Delta(\cdot|\omega^{\beta})$. In particular,

$$\frac{\delta\beta\omega^{\beta}}{\delta\beta\omega^{\beta}-\alpha}p^{\beta} \leq p^*. \tag{OA.5}$$

Rearranging, we have

$$p^{\beta} + \frac{1}{\overline{D}(p^{\beta})}\int_{p^{\beta}}^{\infty}\overline{D}(v)\,\mathrm{d}v \leq p^*,$$

which in turn implies, by (OA.4) and (OA.5),

$$\begin{aligned} p^* - p^{\beta} \geq & \frac{1}{\overline{D}(p^{\beta})}\int_{p^{\beta}}^{\infty}\overline{D}(v)\,\mathrm{d}v \\ = & \frac{1}{\overline{D}(p^{\beta})}\frac{1-\gamma\delta}{\delta\beta} \\ \geq & \frac{1-\gamma\delta}{\delta\beta}\overline{D}(v^{-1}(p^*)). \end{aligned}$$

Therefore,

$$p^* - \frac{1-\gamma\delta}{\delta\beta}\overline{D}(v^{-1}(p^*)) \geq p^{\beta}.$$

Together with the definition of ω^*, we have

$$\int_{\left(1-\frac{\alpha}{\delta\beta\omega^*}\right)p^*}^{\infty}\overline{D}(v)\,\mathrm{d}v = \int_{p^*-\frac{1-\gamma\delta}{\delta\beta}\overline{D}(v^{-1}(p^*))}^{\infty}\overline{D}(v)\,\mathrm{d}v \leq \int_{p^{\beta}}^{\infty}\overline{D}(v)\,\mathrm{d}v = \frac{1-\gamma\delta}{\delta\beta}$$

and hence

$$\delta\left(\gamma+\beta\int_{\left(1-\frac{\alpha}{\delta\beta\omega^*}\right)p^*}^{\infty}\overline{D}(v)\,\mathrm{d}v\right)\leq 1.$$

Thus, $h(\omega^*)\leq\omega^*$, which implies that $\underline{\omega}=\omega^*$.

Next, for any $\omega\in[\underline{\omega},\omega^{\beta}]$ and for any $p\geq 0$, let $\Lambda^{\beta}(p,\omega)$ be the value of the constraint maximization problem

$$\sup_{D\in\mathcal{D}}\left[\alpha pD(p)+\delta\left(\gamma+\beta\int_{p}^{\infty}D(v)\,\mathrm{d}v\right)u\left(\frac{1-\gamma\delta}{\alpha}\omega\right)\right] \tag{OA.6}$$

$$\text{s.t. } \frac{\alpha pD(p)}{1-\gamma\delta}\leq\omega, \tag{OA.7}$$

where u is defined in (OA.1). Let $\underline{\boldsymbol{\omega}}^f(\beta)$ be defined as

$$\underline{\boldsymbol{\omega}}^f(\beta):=\inf\{\omega\in[\underline{\omega},\omega^{\beta}]|\Lambda^{\beta}(\mathbb{E}[v],\omega)\geq\alpha\mathbb{E}[v]+\gamma\delta\omega\}.$$

By the same arguments as in **Case 1**, $\underline{\boldsymbol{\omega}}^f(\beta)\geq\underline{\omega}$ is well-defined and $\Lambda^{\beta}(p,\omega)\geq\alpha\mathbb{E}[v]+\gamma\delta\omega$ if and only if $\omega\in[\underline{\boldsymbol{\omega}}^f(\beta),\omega^{\beta}]$.

Also, let

$$\overline{\boldsymbol{\omega}}^f(\beta):=u\left(\frac{1-\gamma\delta}{\alpha}\underline{\boldsymbol{\omega}}^f(\beta)\right).$$

We now argue that in any subgame perfect equilibrium, the total revenue in each period must be at least $\underline{r}:=(1-\gamma\delta)\underline{\boldsymbol{\omega}}^f(\beta)/\alpha$. Indeed, suppose the contrary, that there exists a subgame perfect equilibrium in which the lowest total revenue on the equilibrium path, say $\tilde{r}$, is strictly below $\underline{r}$. Then, since the seller must be best responding in the period where the total revenue is $\tilde{r}$, for any $p\geq 0$, there must exists $D\in\mathcal{D}$ such that

$$pD(p)\leq\tilde{r},$$

so that the intermediary would respond by choosing $D\in\mathcal{D}$ if the seller deviates to any price $p\geq 0$. Let $\omega(p,D)$ denote the continuation value at this history. In the meantime, at any such history, since it is always feasible for the intermediary to choose the myopically optimal demand, and since her continuation payoff must be at least α share of the present discounted value of the sum of total revenues onward, which in turn, by hypothesis, is no less than $\tilde{r}$, it must be that

$$\alpha pD(p)+\delta\left(\gamma+\beta\int_{p}^{\infty}D(v)\,\mathrm{d}v\right)u(\tilde{r})\geq\alpha pD(p)+\delta\omega(p,D)\geq\alpha p\overline{D}(v^{-1}(p))+\gamma\delta\frac{\alpha\tilde{r}}{1-\gamma\delta},$$

where the first inequality follows from Lemma OA.3 since the total revenue in any period is at least $\tilde{r}$. Therefore, let $\tilde{\omega}:=\alpha\tilde{r}/(1-\gamma\delta)$, it follows that for any $p\geq 0$, there exists D such that

$$\alpha pD(p)+\delta\left(\gamma+\beta\int_{p}^{\infty}D(v)\,\mathrm{d}v\right)u\left(\frac{1-\gamma\delta}{\alpha}\tilde{\omega}\right)\geq\alpha p\overline{D}(v^{-1}(p))+\gamma\delta\tilde{\omega}$$

and that

$$\frac{\alpha pD(p)}{1-\gamma\delta}\leq\tilde{\omega}$$

and hence

$$\Lambda^{\beta}(p,\tilde{\omega})\geq\alpha p\overline{D}(v^{-1}(p))+\gamma\delta\tilde{\omega}$$

for all $p \geq 0$, but $\tilde{\omega} < \underline{\boldsymbol{\omega}}^f(\beta)$, which contradicts to the definition of $\underline{\boldsymbol{\omega}}^f(\beta)$. Thus, in any subgame perfect equilibrium, the total revenue in each period must be at least $\underline{r} := (1-\gamma\delta)\underline{\boldsymbol{\omega}}^f(\beta)/\alpha$. This in turn implies that the intermediary's continuation payoff at any history must be at least

$$\frac{\alpha \underline{r}}{1-\gamma\delta} = \underline{\boldsymbol{\omega}}^f(\beta).$$

Furthermore, by Lemma OA.3, the intermediary's equilibrium payoff must be below $\overline{\boldsymbol{\omega}}^f(\beta)$. Together, the intermediary's equilibrium payoff must be at least $\underline{\boldsymbol{\omega}}^f(\beta)$ and at most $\overline{\boldsymbol{\omega}}^f(\beta)$.

Consider any $\hat{\omega} \in [\underline{\boldsymbol{\omega}}^f(\beta), \omega^\beta]$. By continuity of the function $p \mapsto p\overline{D}(v^{-1}(p))$, since $\alpha\mathbb{E}[v]/(1-\gamma\delta) \geq \omega^\beta$, there exists $\hat{p}, \underline{p}$ such that $\hat{p} \leq \underline{p} \leq p^*$ and that $\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))/(1-\gamma\delta) = \hat{\omega}$, $\alpha\underline{p}\overline{D}(v^{-1}(\underline{p}))/(1-\gamma\delta) = \underline{\boldsymbol{\omega}}^f(\beta)$.

Notice that since $\hat{\omega} \leq \omega^\beta \leq \alpha\mathbb{E}[v]/(1-\gamma\delta)$,[1] there exists $\tilde{p}^\beta \in [\mathbb{E}[v], \hat{p}]$ such that $\omega^\beta = \alpha\tilde{p}^\beta\overline{D}(v^{-1}(\tilde{p}^\beta))/(1-\gamma\delta)$. Moreover, since $\omega^\beta = g^\beta(p^\beta)$,

$$p^\beta = v^{-1}\left(\frac{\delta\beta\omega^\beta}{\delta\beta\omega^\beta - \alpha}p^\beta\right),$$

and hence

$$v(p^\beta) = \frac{\delta\beta\omega^\beta}{\delta\beta\omega^\delta_\beta - \alpha}p^\beta = \tilde{p}^\beta \leq \hat{p}.$$

Rearranging, we have

$$\hat{p} - p^\beta \geq \frac{1}{\overline{D}(p^\beta)}\int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v = \frac{1}{\overline{D}(p^\beta)}\frac{1-\gamma\delta}{\delta\beta} \geq \frac{1-\gamma\delta}{\delta\beta}\overline{D}(v^{-1}(\hat{p})),$$

and hence

$$\left(1 - \frac{\alpha}{\delta\beta\hat{\omega}}\right)\hat{p} \geq p^\beta,$$

which in turn implies that

$$\xi(\hat{p}|\hat{\omega}) = v^{-1}(\hat{p}) \tag{OA.8}$$

For any $p \geq 0$, fix any solution of (OA.6) with $\omega = \underline{\boldsymbol{\omega}}^f(\beta)$ and denote it by D_p. We now construct a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$.

- Start by playing regime $\hat{p}$-MYOPIC. If the seller deviates to $p' \neq \hat{p}$, then enter regime $D_{p'}$-PUNISH immediately. If the intermediary deviates, then move to regime $\underline{p}$-MYOPIC. Otherwise, stay in the same regime.
- Under regime $D_{p'}$-PUNISH, if the intermediary deviates, then move to regime $\underline{p}$-MYOPIC. Otherwise, move to regime $(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)-\alpha), \overline{\boldsymbol{\omega}}^f(\beta))$-TRANSITION.

[1] The second inequality follows from the definition of p^β, which implies

$$\begin{aligned}
\omega^\beta = g^\beta(p^\beta) =& \frac{\alpha}{\delta\beta}\left(1 + \frac{p^\beta\overline{D}(p^\beta)}{\int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v}\right) \\
=& \frac{\alpha}{\delta\beta}\frac{\delta\beta}{1-\gamma\delta}\left(p^\beta\overline{D}(p^\beta) + \int_{p^\beta}^\infty \overline{D}(v)\,\mathrm{d}v\right) \\
\leq& \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta}.
\end{aligned}$$

- Under regime $\underline{p}$-MYOPIC, if the seller deviates to $p' \neq \underline{p}$, then enter regime $D_{p'}$-PUNISH immediately. Otherwise, stay in the same regime.
- Under regime $(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)-\alpha), \overline{\boldsymbol{\omega}}^f(\beta))$-TRANSITION, if the seller deviates to any $p' \geq 0$, move to $D_{p'}$-PUNISH, otherwise, stay in the same regime.

To see that this is a subgame perfect equilibrium, first note that the intermediary's continuation payoff induced by this strategy profile is finite in every subgame. As a result, Lemma 1 implies that it suffices to show there are no incentives to deviate in each regime. Indeed, under regime $(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)-\alpha), \overline{\boldsymbol{\omega}}^f(\beta))$-TRANSITION. First notice that if all players follow their strategies, by Lemma 3, since $\overline{\boldsymbol{\omega}}^f(\beta) \geq \omega^\beta = g^\beta(p^\beta)$,

$$p^\beta \geq v^{-1}\left(\frac{\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)}{\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)-\alpha}\right)$$

and hence the intermediary's payoff would be

$$\delta\left(\gamma+\beta\int_{p^\beta}^{\infty}\overline{D}(v)\,\mathrm{d}v\right)\overline{\boldsymbol{\omega}}^f(\beta) = \overline{\boldsymbol{\omega}}^f(\beta).$$

Moreover, since the intermediary chooses $D \in \Delta(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)-\alpha)|\overline{\boldsymbol{\omega}}^f(\beta))$, she does not have any incentives to deviate. In the meantime, given the intermediary's strategy, if the seller follows his strategy, his payoff would be

$$(1-\alpha)\frac{\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)}{\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)-\alpha}p^\beta\overline{D}(p^\beta),$$

while if he deviates, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is a solution of (OA.6) with $\omega = \underline{\boldsymbol{\omega}}^f(\beta)$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\boldsymbol{\omega}}^f(\beta) = \frac{\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)}{\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)-\alpha}p^\beta\overline{D}(p^\beta),$$

where the equality follows from the definition of $\overline{\boldsymbol{\omega}}^f(\beta)$. Thus, the seller does not have any incentive to deviate either.

In the meantime, for $p \in \{\hat{p}, \underline{p}\}$, under regime p-myopic, if all players follow their strategies, the seller's payoff would be $(1-\alpha)p\overline{D}(v^{-1}(p))$ and the intermediary's payoff would be $\alpha p\overline{D}(v^{-1}(p))/(1-\gamma\delta)$. Meanwhile, if the seller deviates to any $p' \geq 0$, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is a solution of (OA.6) with $\omega = \underline{\boldsymbol{\omega}}^f(\beta)$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\boldsymbol{\omega}}^f(\beta) = \underline{p}\overline{D}(v^{-1}(\underline{p})) \leq p\overline{D}(v^{-1}(p)).$$

Thus, the seller does not have an incentive to deviate. As for the intermediary, if she deviates from this strategy, her continuation value would be at most $\underline{\boldsymbol{\omega}}^f(\beta)$. Therefore, since

$$\frac{\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))}{1-\gamma\delta} = \alpha\hat{p}\overline{D}(v^{-1}(p)) + \gamma\delta\hat{\omega} \geq \alpha\hat{p}\overline{D}(v^{-1}(\hat{p})) + \gamma\delta\underline{\boldsymbol{\omega}}^f(\beta)$$

and

$$\frac{\alpha\underline{p}\overline{D}(v^{-1}(\underline{p}))}{1-\gamma\delta} = \alpha\underline{p}\overline{D}(v^{-1}(\underline{p})) + \gamma\delta\underline{\boldsymbol{\omega}}^f(\beta),$$

by (OA.8), the intermediary's payoff from deviation is at most

$$\sup_{D\in\mathcal{D}}\left[\alpha pD(p)+\delta\left(\gamma+\beta\int_p^\infty D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^f(\beta)\right]=\alpha p\overline{D}(v^{-1}(p))+\gamma\delta\underline{\boldsymbol{\omega}}^f(\beta)$$

and hence the intermediary does not have an incentive to deviate either.

Lastly, for any $p'\geq 0$, under regime $D_{p'}$-PUNISH, if the intermediary follows the strategy, her continuation payoff would be $\overline{\boldsymbol{\omega}}^f(\beta)$, whereas if he deviates, her payoff would be at most $\alpha p'\overline{D}(v^{-1}(p'))+\gamma\delta\underline{\boldsymbol{\omega}}^f(\beta)$. Since $D_{p'}$ is a solution of (OA.6) with $\omega=\underline{\boldsymbol{\omega}}^f(\beta)$, it must be that

$$\alpha p' D_{p'}(p')+\delta\left(\gamma+\beta\int_{p'}^\infty D_{p'}(v)\,\mathrm{d}v\right)\overline{\boldsymbol{\omega}}^f(\beta)=\Lambda^\beta(p',\underline{\boldsymbol{\omega}}^f(\beta))\geq\alpha p'\overline{D}(v^{-1}(p'))+\gamma\delta\underline{\boldsymbol{\omega}}^f(\beta),$$

where the last inequality follows from the definition of $\underline{\boldsymbol{\omega}}^f(\beta)$. Thus, the intermediary does not have any incentives to deviate under this regime.

Therefore, the strategy profile above is indeed a subgame perfect equilibrium. Moreover, the intermediary's payoff in this equilibrium is

$$\frac{\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))}{1-\gamma\delta}=\hat{\omega}.$$

Now consider any $\hat{\omega}\in[\omega^\beta,\overline{\boldsymbol{\omega}}^f(\beta)]$. We claim that the following strategy profile consititutes a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$.

- Start by playing regime $(\delta\beta\hat{\omega}p^\beta/(\delta\beta\hat{\omega}-\alpha),\hat{\omega})$-TRANSITION. If the seller deviates to any $p'\geq 0$, move to regime p'-PUNISH. Otherwise, stay in the same regime.

- Under regime $D_{p'}$-PUNISH, if the intermediary deviates, then move to regime $\underline{p}$-MYOPIC. Otherwise, move to regime $(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)-\alpha),\overline{\boldsymbol{\omega}}^f(\beta))$-TRANSITION.

- Under regime $\underline{p}$-MYOPIC, if the seller deviates to $p'\neq\underline{p}$, then enter regime $D_{p'}$-PUNISH immediately. Otherwise, stay in the same regime.

- Under regime $(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)-\alpha),\overline{\boldsymbol{\omega}}^f(\beta))$-TRANSITION, if the seller deviates to any $p'\geq 0$, move to $D_{p'}$-PUNISH, otherwise, stay in the same regime.

From the same arguments as above, it follows that both the intermediary and the seller do not have incentives to deviate under regime p'-PUNISH, regime $\underline{p}$-MYOPIC, and regime $(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)p^\beta/(\delta\beta\overline{\boldsymbol{\omega}}^f(\beta)-\alpha),\overline{\boldsymbol{\omega}}^f(\beta))$-TRANSITION. Therefore, by Lemma 1, since the intermediary's continuation payoff are finite in all subgames under this strategy profile, it suffices to show that both the intermediary and the seller do not have incentives to deviate under regime $(\delta\beta\hat{\omega}p^\beta/(\delta\beta\hat{\omega}-\alpha),\hat{\omega})$-TRANSITION and that the intermediary's equilibrium payoff is $\hat{\omega}$. Indeed, since $\hat{\omega}\geq\omega^\beta=g^\beta(p^\beta)$,

$$p^\beta\geq v^{-1}\left(\frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^\beta\right)$$

and hence by Lemma 3, the intermediary's payoff would be

$$\delta\left(\gamma+\beta\int_{p^\beta}^\infty\overline{D}(v)\,\mathrm{d}v\right)\hat{\omega}=\hat{\omega}.$$

Moreover, since the intermediary chooses $D \in \Delta(\delta\beta\hat{\omega}p^{\beta}/(\delta\beta\hat{\omega}-\alpha)|\hat{\omega})$, she does not have any incentives to deviate. In the meantime, given the intermediary's strategy, if the seller follows his strategy, his payoff would be

$$(1-\alpha)\frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^{\beta}\overline{D}(p^{\beta}),$$

while if he deviates, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is a solution of (OA.6) with $\omega = \underline{\boldsymbol{\omega}}^f(\beta)$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\boldsymbol{\omega}}^f(\beta) \leq \frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^{\beta}\overline{D}(p^{\beta}),$$

where the second inequality follows from $\hat{\omega} \leq \overline{\boldsymbol{\omega}}^f(\beta)$ and from the definition of $\overline{\boldsymbol{\omega}}^f(\beta)$. Thus, the seller does not have any incentive to deviate either, as desired.

Case 3: $\beta \in [\widehat{\beta}, \overline{\beta}]$.
In this case, using the same argument as in **Case 2**, it follows that $\underline{\omega} = \omega^*$ as well. Let $\underline{\boldsymbol{\omega}}^f(\beta) := \underline{\omega}$ and let $\overline{\boldsymbol{\omega}}^f(\beta) = \infty$. For any $\hat{\omega} \geq \underline{\omega}$, we will construct a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$. First, consider any $\hat{\omega} \in [\underline{\omega}, \omega^{\beta}]$. Since $\omega^{\beta} \leq \alpha\mathbb{E}[v]/(1-\gamma\delta)$,[2] there exists $\hat{p} \in [\mathbb{E}[v], p^*]$ such that $\hat{p}\overline{D}(v^{-1}(\hat{p})) = (1-\gamma\delta)\hat{\omega}/\alpha$. Moreover, as shown in **Case 2**, it must be that

$$\left(1 - \frac{\alpha}{\delta\beta\hat{\omega}}\right)\hat{p} \geq p^{\beta}$$

and therefore

$$\xi(\hat{p}|\hat{\omega}) = v^{-1}(\hat{p}). \tag{OA.9}$$

We now construct a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$. To this end, notice that by definition of p^*, for any p', there exists $D_{p'} \in \mathcal{D}$ such that $p'D_{p'}(p'^{+}) \leq p^*\overline{D}(v^{-1}(p^*))$. For any $p' \geq 0$, fix any such $D_{p'} \in \mathcal{D}$. In the meantime, take any $\tilde{\omega} > \max\{\alpha\mathbb{E}[v]/\gamma\delta + \underline{\omega}, \omega^{\beta}\}$. Now consider the following strategy profile:

- Start by playing regime $\hat{p}$-MYOPIC. If the seller deviates to any $p' \neq \hat{p}$, move to $D_{p'}$-PUNISH. If the intermediary deviates, move to regime p^*-MYOPIC. Otherwise, stay in the same regime.
- Under regime $D_{p'}$-PUNISH, if the intermediary deviates, move to regime p^*-MYOPIC. Otherwise, move to regime $(\delta\beta\tilde{\omega}p^{\beta}/(\delta\beta\tilde{\omega}-\alpha), \tilde{\omega})$-TRANSITION.
- Under regime $(\delta\beta\tilde{\omega}p^{\beta}/(\delta\beta\tilde{\omega}-\alpha), \tilde{\omega})$-TRANSITION, if the seller deviates to $p' \geq 0$, move to regime $D_{p'}$-PUNISH. Otherwise, stay in the same regime.
- Under regime p^*-MYOPIC, if the seller deviates to any $p' \neq p^*$, move to $D_{p'}$-PUNISH. Otherwise, stay in the same regime.

We claim that the strategy profile above constitutes a subgame perfect equilibrium and that the intermediary's payoff is $\hat{\omega}$. To see this, first note since the intermediary's continuation induced by this strategy profile in every subgame is finite, by Lemma 1, it suffices to show that both the intermediary and the

[2] See footnote 1.

seller do not have incentives to deviate under each of the regimes above, given that the other player plays according to this strategy.

Under regime $(\delta\beta\tilde{\omega}p^{\beta}/(\delta\beta\tilde{\omega}-\alpha),\tilde{\omega})$-TRANSITION. First notice that if all players follow their strategies, by Lemma 3, since $\tilde{\omega} \geq \omega^{\beta} = g^{\beta}(p^{\beta})$,

$$p^{\beta} \geq v^{-1}\left(\frac{\delta\beta\omega^{\beta}}{\delta\beta\omega^{\beta}-\alpha}\right)$$

and hence the intermediary's payoff would be

$$\delta\left(\gamma+\beta\int_{p^{\beta}}^{\infty}\overline{D}(v)\,\mathrm{d}v\right)\tilde{\omega}=\tilde{\omega}.$$

Moreover, since the intermediary chooses $D \in \Delta(\delta\beta\tilde{\omega}p^{\beta}/(\delta\beta\tilde{\omega}-\alpha)|\tilde{\omega})$, she does not have any incentives to deviate. In the meantime, given the intermediary's strategy, if the seller follows his strategy, his payoff would be

$$(1-\alpha)\frac{\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega}-\alpha}p^{\beta}\overline{D}(p^{\beta}),$$

while if he deviates, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is chosen so that $p'D_{p'}(p') \leq p^*\overline{D}(v^{-1}(p^*))$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\omega} \leq p^{\beta}\overline{D}(p^{\beta}) \leq \frac{\delta\beta\tilde{\omega}}{\delta\beta\tilde{\omega}-\alpha}p^{\beta}\overline{D}(p^{\beta}),$$

where the first inequality follows from $\beta \geq \widehat{\beta}$, which in turn implies that $p^{\beta}\overline{D}(p^{\beta}) \geq p^*\overline{D}(v^{-1}(p^*))$. Thus, the seller does not have any incentive to deviate either.

In the meantime, for $p \in \{\hat{p}, p^*\}$, under regime p-MYOPIC, if all players follow their strategies, the seller's payoff would be $(1-\alpha)p\overline{D}(v^{-1}(p))$ and the intermediary's payoff would be $\alpha p\overline{D}(v^{-1}(p))/(1-\gamma\delta)$. Meanwhile, if the seller deviates to any $p' \geq 0$, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is chosen so that $p'D_{p'}(p') \leq p^*\overline{D}(v^{-1}(p^*))$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\omega} = p^*\overline{D}(v^{-1}(p^*)) \leq p\overline{D}(v^{-1}(p)).$$

Thus, the seller does not have an incentive to deviate. As for the intermediary, if she deviates from this strategy, her continuation value would be at most $\underline{\omega}$. Therefore, since

$$\frac{\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))}{1-\gamma\delta} = \alpha\hat{p}\overline{D}(v^{-1}(p)) + \gamma\delta\hat{\omega} \geq \alpha\hat{p}\overline{D}(v^{-1}(\hat{p})) + \gamma\delta\underline{\omega}$$

and

$$\frac{\alpha\underline{p}\overline{D}(v^{-1}(\underline{p}))}{1-\gamma\delta} = \alpha p^*\overline{D}(v^{-1}(p^*)) + \gamma\delta\underline{\omega},$$

by (OA.9), the intermediary's payoff from deviation is at most

$$\begin{aligned}&\sup_{D\in\mathcal{D}}\left[\alpha pD(p)+\delta\left(\gamma+\beta\int_{p}^{\infty}D(v)\,\mathrm{d}v\right)\underline{\omega}\right]\\=&\alpha p\overline{D}(v^{-1}(p))+\gamma\delta\underline{\boldsymbol{\omega}}^{f}(\beta)\end{aligned}$$

and hence the intermediary does not have an incentive to deviate either.

Lastly, for any $p' \geq 0$, under regime $D_{p'}$-PUNISH, if the intermediary follows the strategy, her continuation payoff would be $\tilde{\omega}$, whereas if she deviates, her payoff would be at most $\alpha p' \overline{D}(v^{-1}(p')) + \gamma\delta\underline{\omega}$. Since $\tilde{\omega} > \alpha\mathbb{E}[v]/\gamma\delta + \underline{\omega}$, it must be that

$$\alpha p' D_{p'}(p') + \delta\left(\gamma + \beta \int_{p'}^{\infty} D_{p'}(v)\,\mathrm{d}v\right)\tilde{\omega} \geq \gamma\delta\tilde{\omega} > \alpha\mathbb{E}[v] + \gamma\delta\underline{\omega} \geq \alpha p'\overline{D}(v^{-1}(p')) + \gamma\delta\underline{\omega}.$$

Thus, the intermediary does not have any incentives to deviate under this regime.

Therefore, the strategy profile above is indeed a subgame perfect equilibrium. Moreover, the intermediary's payoff in this equilibrium is

$$\frac{\alpha\hat{p}\overline{D}(v^{-1}(\hat{p}))}{1-\gamma\delta} = \hat{\omega}.$$

Next, consider any $\hat{\omega} > \omega^{\beta}$. Again, take any $\tilde{\omega} > \max\{\alpha\mathbb{E}[v]/\gamma\delta, \omega^{\beta}\}$, we claim that the following strategy profile is a subgame perfect equilibrium and the intermediary's payoff is $\hat{\omega}$.

- Start by playing regime $(\delta\beta\hat{\omega}p^{\beta}/(\delta\beta\hat{\omega}-\alpha), \hat{\omega})$-TRANSITION. If the seller deviates to any $p' \geq 0$, move to regime p'-PUNISH. Otherwise, stay in the same regime.

- Under regime $D_{p'}$-PUNISH, if the intermediary deviates, then move to regime p^*-MYOPIC. Otherwise, move to regime $(\delta\beta\tilde{\omega}p^{\beta}/(\delta\beta\tilde{\omega}-\alpha), \tilde{\omega})$-TRANSITION.

- Under regime p^*-MYOPIC, if the seller deviates to $p' \neq p^*$, then enter regime $D_{p'}$-PUNISH immediately. Otherwise, stay in the same regime.

- Under regime $(\delta\beta\tilde{\omega}(\alpha,\beta,\delta)p^{\beta}/(\delta\beta\tilde{\omega}-\alpha), \tilde{\omega})$-TRANSITION, if the seller deviates to any $p' \geq 0$, move to $D_{p'}$-PUNISH, otherwise, stay in the same regime.

From the same arguments as above, it follows that both the intermediary and the seller do not have incentives to deviate under regime p'-PUNISH, regime p^*-MYOPIC, and regime $(\delta\beta\tilde{\omega}p^{\beta}/(\delta\beta\tilde{\omega}-\alpha), \tilde{\omega})$-TRANSITION. Therefore, by Lemma 1, since the intermediary's continuation payoff in any subgame is finite under this strategy profile, it suffices to show that both the intermediary and the seller do not have incentives to deviate under regime $(\delta\beta\hat{\omega}p^{\beta}/(\delta\beta\hat{\omega}-\alpha), \hat{\omega})$-TRANSITION and that the intermediary's equilibrium payoff is $\hat{\omega}$. Indeed, since $\hat{\omega} \geq \omega^{\beta} = g^{\beta}(p^{\beta})$,

$$p^{\beta} \geq v^{-1}\left(\frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^{\beta}\right)$$

and hence by Lemma 3, the intermediary's payoff would be

$$\delta\left(\gamma + \beta\int_{p^{\beta}}^{\infty}\overline{D}(v)\,\mathrm{d}v\right)\hat{\omega} = \hat{\omega}.$$

Moreover, since the intermediary chooses $D \in \Delta(\delta\beta\hat{\omega}p^{\beta}/(\delta\beta\hat{\omega}-\alpha)|\hat{\omega})$, she does not have any incentives to deviate. In the meantime, given the intermediary's strategy, if the seller follows his strategy, his payoff would be

$$(1-\alpha)\frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^{\beta}\overline{D}(p^{\beta}),$$

while if he deviates, his payoff would be $(1-\alpha)p'D_{p'}(p')$. Since $D_{p'}$ is chosen so that $p'D_{p'}(p') \leq \underline{\omega}$, it must be that

$$p'D_{p'}(p') \leq \frac{1-\gamma\delta}{\alpha}\underline{\omega} \leq p^{\beta}\overline{D}(p^{\beta})\frac{\delta\beta\hat{\omega}}{\delta\beta\hat{\omega}-\alpha}p^{\beta}\overline{D}(p^{\beta}),$$

where the second inequality follows from the fact that $\beta \geq \widehat{\beta}$ and the definition of $\widehat{\beta}$. Thus, the seller does not have any incentive to deviate either, as desired.

Case 4: $\beta \in (\overline{\beta}, \beta^*)$.
If $\underline{\omega} = \omega^*$, then the argument of **Case 3** applies. Therefore, it suffices to consider the case where $\underline{\omega} > \omega^*$. In this case, it must be that

$$\delta \left(\gamma + \beta \int_{\left(1-\frac{\alpha}{\beta\delta\underline{\omega}}\right)p^*}^{\infty} \overline{D}(v)\,\mathrm{d}v \right) = 1 \iff h(\underline{\omega}) = \underline{\omega},$$

which also implies that $\underline{\omega} > \alpha/\delta\beta$. Furthermore, by Lemma OA.1, since $\mathbf{W}(p^*|\underline{\omega}) = \underline{\omega}$ and since

$$\delta \left(\gamma + \beta \int_{\left(1-\frac{\alpha}{\delta\beta\underline{\omega}}\right)p^*}^{\infty} \overline{D}(v)\,\mathrm{d}v \right) = 1,$$

we have

$$\underline{\omega} = \mathbf{W}(p^*|\underline{\omega}) \geq \alpha p^* \overline{D}(v^{-1}(p^*)) + \gamma\delta\underline{\omega},$$

and hence

$$\delta\beta\underline{\omega} \int_{\left(1-\frac{\alpha}{\delta\beta\underline{\omega}}\right)p^*}^{\infty} \overline{D}(v)\,\mathrm{d}v \geq \alpha p^* \overline{D}(v^{-1}(p^*)),$$

which in turn is equivalent to

$$\left(1 - \frac{\alpha}{\delta\beta\underline{\omega}}\right) p^* \geq v^{-1}(p^*). \tag{OA.10}$$

In the meantime, notice that for any $\omega \geq \underline{\omega}$, since $p \mapsto p\overline{D}((1-\alpha/\delta\beta\omega)p)$ is quasi-concave,

$$\overline{p}\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega_{t+1}}\overline{p}\right)\right) > \overline{p}\overline{D}(\overline{p}) \geq p^*\overline{D}(v^{-1}(p^*)),$$

where the last inequality follows from the definition of p^*. Moreover, for any $\omega \geq \underline{\omega}$, by (OA.10),

$$p^*\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)p^*\right) < p^*\overline{D}(v^{-1}(p^*)),$$

and hence there exists a unique $\boldsymbol{p}(\omega) \in [\overline{p}, p^*]$ such that

$$\boldsymbol{p}(\omega)\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)\boldsymbol{p}(\omega)\right) = p^*\overline{D}(v^{-1}(p^*)). \tag{OA.11}$$

By definition, the function $\boldsymbol{p}$ is continuous, bounded by $\overline{p}$ and p^*, and such that $\omega \mapsto (1-\alpha/\beta\delta\omega)\boldsymbol{p}(\omega)$ is decreasing. Define

$$\overline{\omega} := \delta\left(\gamma + \beta\int_{\left(1-\frac{\alpha}{\delta\beta\underline{\omega}}\right)\boldsymbol{p}(\underline{\omega})}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\underline{\omega}.$$

It then follows that $\overline{\omega} > \underline{\omega}$.

We now introduce a recursive formula.

Algorithm 1. For any $t \in \mathbb{N} \cup \{0\}$, given any $\omega_t \geq \overline{\omega}$, take ω_{t+1} so that[3]

$$\omega_t = \left(\gamma + \beta \int_{\left(1 - \frac{\alpha}{\delta \beta \omega_{t+1}}\right) \boldsymbol{p}(\omega_{t+1})}^{\infty} \overline{D}(v) \, \mathrm{d}v \right) \omega_{t+1}.$$

Then, let $p_t := \boldsymbol{p}(\omega_{t+1})$.

If $\omega_{t+1} \geq \overline{\omega}$, then repeat the procedure by letting $\omega_t = \omega_{t+1}$. Otherwise, stop. ◇

From Algorithm 1, for any initial value $\omega_0 \geq \overline{\omega}$, we may obtain sequences $\{\omega_t\}_{t=0}^{T}$ and $\{p_t\}_{t=0}^{T-1}$ such that

$$\omega_t = \left(\gamma + \beta \int_{\left(1 - \frac{\alpha}{\delta \beta \omega_{t+1}}\right) p_t}^{\infty} \overline{D}(v) \, \mathrm{d}v \right) \omega_{t+1}, \tag{OA.12}$$

that

$$p_t \overline{D} \left(\left(1 - \frac{\alpha}{\delta \beta \omega_{t+1}}\right) p_t \right) = p^* \overline{D}(v^{-1}(p^*)), \tag{OA.13}$$

and that

$$\left(1 - \frac{\alpha}{\delta \beta \omega_{t+1}}\right) p_t \geq v^{-1}(p_t). \tag{OA.14}$$

Consider any $\hat{\omega} \geq \underline{\omega}$. We will now construct subgame perfect equilibria that give the intermediary equilibrium payoff $\hat{\omega}$. First, suppose that $\hat{\omega} \leq \overline{\omega}$. In this case, take any $T \in \mathbb{N}$ such that [4]

$$\frac{1}{\delta^T} \left(\underline{\omega} - \frac{1 - \gamma^{T+1} \delta^{T+1}}{1 - \gamma \delta} \alpha p^* \overline{D}(v^{-1}(p^*)) \right) > \max \left\{ \overline{\omega}, \frac{1}{\delta} \alpha \mathbb{E}[v] + (\gamma + \beta \mathbb{E}[v]) \, \underline{\omega} \right\}.$$

Meanwhile, for any $\omega \in [\underline{\omega}, \overline{\omega}]$, let

$$\phi(\omega) := \frac{1}{\delta^T} \left(\omega - \frac{1 - \gamma^{T+1} \delta^{T+1}}{1 - \gamma \delta} \alpha p^* \overline{D}(v^{-1}(p^*)) \right),$$

and let $\{\boldsymbol{\omega}_t(\omega)\}_{t=0}^{T^\omega}$ and $\{\boldsymbol{p}_t(\omega)\}_{t=0}^{T^\omega - 1}$ denote the sequences obtain from Algorithm 1 with $\phi(\omega)$ being the initial value. Meanwhile, for any $p' \geq 0$, by the definition of p^*, there exists $D_{p'} \in \mathcal{D}$ such that $p' D_{p'}(p'^{+}) \leq p^* \overline{D}(v^{-1}(p^*))$. Fix any such $D_{p'} \in \mathcal{D}$ for all p'. We now describe the desired subgame perfect equilibrium. For expositional convenience, we index strategies by a state variable ω.

- Set the state as $\hat{\omega}$. Start by playing regime p^*-MYOPIC with state $\hat{\omega}$.
- For any state $\omega \in [\underline{\omega}, \overline{\omega}]$, under regime p^*-MYOPIC with state $\omega \in [\underline{\omega}, \overline{\omega}]$, if no one has deviated and if this regime has been played for less than T periods, stay in the same regime and the same state; if the seller deviates to any $p' \neq p^*$, move to regime $D_{p'}$-PUNISH immediately while keeping the state unchanged; if the intermediary deviates, reset the count, set the state to $\underline{\omega}$, and stay under the same regime. Otherwise, keep the state unchanged and move to regime $(\boldsymbol{p}_0(\omega), \boldsymbol{\omega}_1(\omega))$-TRANSITION.

[3] Such ω_{t+1} exists since $\omega_t \geq \overline{\omega}$ and

$$\lim_{\omega \to \infty} \delta \left(\gamma + \beta \int_{\left(1 - \frac{\alpha}{\delta \beta \omega}\right) \boldsymbol{p}(\omega)}^{\infty} \overline{D}(v) \, \mathrm{d}v \right) \omega = \infty.$$

[4] Such T exists because

$$\underline{\omega} > \omega^* = \frac{\alpha p^* \overline{D}(v^{-1}(p^*))}{1 - \gamma \delta}.$$

- For any state $\omega \in [\underline{\omega}, \overline{\omega}]$, under regime $(\boldsymbol{p}_{t-1}(\omega), \boldsymbol{\omega}_t(\omega))$-TRANSITION, if the seller deviates to any $p' \neq p^*$, move to regime $D_{p'}$-PUNISH immediately and set the state to $\underline{\omega}$. Otherwise, move to regime $(\boldsymbol{p}_t(\omega), \boldsymbol{\omega}_{t+1}(\omega))$-TRANSITION while keeping the state unchanged in the next period if $t < T^{\omega} - 1$, and move to regime p^*-MYOPIC while setting the state as $\boldsymbol{\omega}_T(\omega)$ if $t = T^{\omega} - 1$.

- Under regime $D_{p'}$-PUNISH with any state $\omega \in [\underline{\omega}, \overline{\omega}]$, if the intermediary deviates, then move to regime p^*-MYOPIC while setting the state as $\underline{\omega}$. Otherwise, set the state to $\overline{\omega}$ and move to regime $(\boldsymbol{p}_0(\overline{\omega}), \boldsymbol{\omega}_0(\overline{\omega}))$-TRANSITION.

To see that this constitutes a subgame perfect equilibrium, first note that the intermediary's continuation payoff in every subgame is finite under this strategy profile. Thus, by Lemma 1, it suffices to show that there are no incentives for one-shot deviations for each player.

To see this, under regime p^*-MYOPIC with any state $\omega \in [\underline{\omega}, \overline{\omega}]$, given that the seller does not deviate, if the intermediary follows the strategy, her present discounted payoff in each period would be at least $\hat{\omega}$.[5] Alternatively, if the intermediary deviates in any period and maintain her continuation strategy, since the continuation play will enter into regime p^*-MYOPIC with state being $\underline{\omega}$, her continuation value would be $\underline{\omega}$. Thus, the best present discounted value she can obtain would be

$$\sup_{D \in \mathcal{D}} \left[\alpha p^* D(p^*) + \delta \left(\gamma + \beta \int_{p^*}^{\infty} \overline{D}(v)\, \mathrm{d}v \right) \underline{\omega} \right] = \underline{\omega},$$

where the equality follows from $h(\underline{\omega}) = \underline{\omega}$. Therefore, the intermediary does not have profitable one-shot deviation under regime p^*-MYOPIC with any state $\omega \in [\underline{\omega}, \overline{\omega}]$. Meanwhile, if the seller follows his strategy, his payoff would be $(1-\alpha)p^*\overline{D}(v^{-1}(p^*))$, while if he deviate to any $p' \geq 0$, his payoff would be $(1-\alpha)p' D_{p'}(p') \leq (1-\alpha)p^*\overline{D}(v^{-1}(p^*))$.

Moreover, for any state $\omega \in [\underline{\omega}, \overline{\omega}]$ and for any $t \in \{0, \ldots, T^{\omega} - 1\}$, under regime-$(\boldsymbol{p}_t(\omega), \boldsymbol{\omega}_{t+1}(\omega))$-TRANSITION, by (OA.12), (OA.14), and Lemma 3,

$$\boldsymbol{\omega}_t(\omega) = \sup_{D \in \mathcal{D}} \left[\alpha \boldsymbol{p}_t(\omega) D(\boldsymbol{p}_t(\omega)) + \delta \left(\gamma + \beta \int_{\boldsymbol{p}_t(\omega)}^{\infty} \overline{D}(v)\, \mathrm{d}v \right) \boldsymbol{\omega}_{t+1}(\omega) \right]$$

and hence the intermediary would not have any incentive to deviate and her payoff in this subgame would be $\boldsymbol{\omega}_t(\omega)$. Meanwhile, if the seller deviates to any $p' \neq \boldsymbol{p}_t(\omega)$, his payoff would be $(1-\alpha)p' D_{p'}(p') \leq (1-\alpha)p^*\overline{D}(v^{-1}(p^*)) = \boldsymbol{p}_t(\omega)\overline{D}((1-\alpha/\delta\beta\omega)\boldsymbol{p}_t(\omega))$, due to (OA.13). Thus, the seller does not have incentives to deviate either.

Lastly, under regime $D_{p'}$-PUNISH with any state $\omega \in [\underline{\omega}, \overline{\omega}]$, if the intermediate deviates, her continuation payoff would be $\underline{\omega}$ and hence the best payoff from deviation is

$$\alpha \mathbb{E}[v] + \delta \left(\gamma + \beta \mathbb{E}[v] \right) \underline{\omega}.$$

[5] This is because the continuation value at the beginning of the $(\boldsymbol{p}_0(\hat{\omega}), \boldsymbol{\omega}_1(\hat{\omega}))$-TRANSITION regime is $\phi(\hat{\omega})$ and since

$$\hat{\omega} = \frac{1 - \gamma^{T+1}\delta^{T+1}}{1 - \gamma\delta} \alpha p^* \overline{D}(v^{-1}(p^*)) + \delta^T \phi(\hat{\omega}) = \sum_{s=0}^{T-1} \gamma^s \delta^s \alpha p^* \overline{D}(v^{-1}(p^*)) + \delta^T \phi(\hat{\omega}).$$

Meanwhile, if she follows the strategy, her continuation payoff would be $\phi(\overline{\omega})$ and hence her payoff would be at least $\delta\phi(\overline{\omega})$. By the definition of T and ϕ,

$$\delta\phi(\overline{\omega}) = \frac{1}{\delta^{T-1}}\left[\overline{\omega} - \frac{1-\gamma^{T+1}\delta^{T+1}}{1-\gamma\delta}\alpha p^*\overline{D}(v^{-1}(p^*))\right] > \alpha\mathbb{E}[v] + \delta\left(\gamma + \beta\mathbb{E}[v]\right)\underline{\omega}.$$

Thus, the intermediary does not have one-shot deviation incentives.

As a result, neither players have profitable one-shot deviations. Moreover, as shown above, since the intermediary' continuation payoff after playing T rounds of p^*-MYOPIC with state $\hat{\omega}$ is exactly $\boldsymbol{\omega}_0(\hat{\omega})$, her on-path payoff under this strategy profile is given by

$$\sum_{t=0}^{T}\gamma^t\delta^t\alpha p^*\overline{D}(v^{-1}(p^*)) + \delta\boldsymbol{\omega}_0(\hat{\omega}) = \frac{1-\gamma^{T+1}\delta^{T+1}}{1-\gamma\delta}\alpha p^*\overline{D}(v^{-1}(p^*)) + \delta\phi(\hat{\omega}) = \hat{\omega} < \infty.$$

Alternatively, if $\hat{\omega} > \overline{\omega}$, we may construct the same type of strategy as follows: Let $\omega_0 := \hat{\omega}$ and let $\{\omega_t\}_{t=0}^{\bar{T}}$ and $\{p_t\}_{t=0}^{\bar{T}-1}$ denote the sequences obtained from Algorithm 1 with the initial value being ω_0. Consider the following strategy profile:

- Start by playing regime (p_0, ω_1)-TRANSITION with a null state $\emptyset$.
- For any $t \in \{1, \ldots, \bar{T}-1\}$, under regime (p_{t-1}, ω_t) with any state, if the seller deviates to any p', move to regime $D_{p'}$-PUNISH immediately while setting the state as $\underline{\omega}$. Otherwise, move to regime (p_t, ω_{t+1})-TRANSITION while keeping the state unchanged if $t < \bar{T}-1$, and move to regime p^*-MYOPIC while setting the state to $\omega_{\bar{T}} \in [\underline{\omega}, \overline{\omega}]$ if $t = \bar{T}-1$.
- For any state ω, under regime p^*-MYOPIC with state $\omega \in [\underline{\omega}, \overline{\omega}]$, if no one has deviated and if this regime has been played for less than T periods, stay in the same regime and the same state; if the seller deviates to any $p' \neq p^*$, move to regime $D_{p'}$-PUNISH immediately while keeping the state unchanged; if the intermediary deviates, reset the count, set the state to $\underline{\omega}$, and stay under the same regime. Otherwise, keep the state unchanged and move to regime $(\boldsymbol{p}_0(\omega), \boldsymbol{\omega}_1(\omega))$-TRANSITION.
- For any state $\omega \in [\underline{\omega}, \overline{\omega}]$, under regime $(\boldsymbol{p}_{t-1}(\omega), \boldsymbol{\omega}_t(\omega))$-TRANSITION, if the seller deviates to any $p' \neq p^*$, move to regime $D_{p'}$-PUNISH immediately and set the state to $\underline{\omega}$. Otherwise, move to regime $(\boldsymbol{p}_t(\omega), \boldsymbol{\omega}_{t+1}(\omega))$-TRANSITION while keeping the state unchanged in the next period if $t < T^{\omega}-1$, and move to regime p^*-MYOPIC while setting the state as $\boldsymbol{\omega}_T(\omega)$ if $t = T^{\omega}-1$.
- Under regime p'-PUNISH with any state $\omega \in [\underline{\omega}, \overline{\omega}]$, if the intermediary deviates, then move to regime p^*-MYOPIC while setting the state as $\underline{\omega}$. Otherwise, set the state to $\overline{\omega}$ and move to regime $(\boldsymbol{p}_0(\overline{\omega}), \boldsymbol{\omega}_0(\overline{\omega}))$-TRANSITION.

By the same arguments as those for the case $\hat{\omega} \in [\underline{\omega}, \overline{\omega}]$, there are no profitable one-shot deviations for all players. Thus, by Lemma 1, since the intermediary's payoff following this strategy profile is $\omega_0 = \hat{\omega}$, this is also a subgame perfect equilibrium.

Together, it follows that whenever $\beta < \beta^*$, there exists $\underline{\omega} \leq \underline{\boldsymbol{\omega}}^f(\beta) \leq \overline{\boldsymbol{\omega}}^f(\beta) \leq \infty$ such that for any $\hat{\omega} \in [\underline{\boldsymbol{\omega}}^f(\beta), \overline{\boldsymbol{\omega}}^f(\beta)]\backslash\{\infty\}$, there exists a finite subgame perfect equiulibrium in which the intermediary's payoff is $\hat{\omega}$. It remains to show that finite subgame perfect equilibria do not exist whenever $\beta \geq \beta^*$. To see this, notice that $\beta \geq \beta^*$ implies $h(\omega) > \omega$ for all $\omega \geq 0$. Now let $\omega_0 := \omega^*$ and define $\{\omega_n\}$ recursively as

$$\omega_n := h(\omega_{n-1}) = \delta \left(\gamma + \beta \int_{\left(1-\frac{\alpha}{\delta\beta\omega_{n-1}}\right)p^*}^{\infty} \overline{D}(v)\,\mathrm{d}v \right) \omega_{n-1},$$

for all $n \in \mathbb{N}$. By the same arguments as in the proof of Lemma 5, the intermediary's continuation payoff at any history in any subgame perfect equilibrium must be at least ω_n for all n. However, since $h(\omega) > \omega$ for all $\omega \geq 0$, $\liminf_{n\to\infty}\{\omega_n\} = \infty$ and hence there is no finite subgame perfect equilibria. This completes the proof. ■

OA.2.3 Proof of Corollary 1

Consider any $\beta \geq 0$ and any finite subgame perfect equilibrium with outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$. Since r^* is the revenue guarantee, $r_t \geq r^*$ for all t. Moreover, by the proof of Theorem 3, $r_t \geq (1-\gamma\delta)\underline{\boldsymbol{\omega}}^f(\beta)/\alpha$ for all t whenever $\beta \leq \overline{\beta}$. Thus, $r_t \geq \underline{r}(\beta)$ for all t. Meanwhile, since $D(0) = 1$ for all $D \in \mathcal{D}$, it must be that $r_t \leq p_t$ for all t.

For any $t \geq 0$, let $D_t \in \mathcal{D}$ be the disclosure policy chosen by the intermediary on the equilibrium path in period t so that

$$\sigma_t = \int_{p_t}^{\infty} D_t(v)\,\mathrm{d}v.$$

Since $D_t \in \mathcal{D}$, $\sigma_t \geq (\mathbb{E}[v] - p_t)^+$. Moreover, since $D_t \in \mathcal{D}$ is nonincreasing, it must be that

$$\int_{p_t}^{\infty} \overline{D}(v)\,\mathrm{d}v \geq \int_{p_t}^{\infty} D_t(v)\,\mathrm{d}v \geq \sigma_t - (p - p_t)D_t(p_t),$$

for all $p \geq 0$. As a result,

$$\sigma_t \leq \min_{p \geq 0} \left[\int_p^{\infty} \overline{D}(v) + (p - p_t)D_t(p_t) \right] = S\left(\frac{r_t}{p_r}\right) - r_t,$$

where the equality follows from the first order condition of the minimization problem, which implies that at the solution $\hat{p}_t$, $\overline{D}(\hat{p}_t) = D_t(p_t) = r_t/p_t$. Together, we have

$$(\mathbb{E}[v] - p_t)^+ \leq \sigma_t \leq S\left(\frac{r_t}{p_r}\right) - r_t,$$

for all $t \geq 0$.

Lastly, for any $t \geq 0$, Theorem 3 implies that $\omega_t \leq \overline{\boldsymbol{\omega}}^f(\beta)$. Moreover, subgame perfection implies that

$$\alpha r_t + \delta(\gamma + \beta\sigma_t)\omega_t \geq \sup_{D \in \mathcal{D}} \left[\alpha p_t + \delta\left(\gamma + \beta\int_{p_t}^{\infty} D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^f(\beta) \right].$$

Together, it must be that

$$\alpha r_t + \delta(\gamma + \beta\sigma_t)\underline{\boldsymbol{\omega}}^f(\beta) \geq \sup_{D \in \mathcal{D}} \left[\alpha p_t + \delta\left(\gamma + \beta\int_{p_t}^{\infty} D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^f(\beta) \right],$$

as desired.

Conversely, given any $(r,\sigma,p) \in \mathbf{Z}^f(\beta)$ and any $T \in \mathbb{N}$, it suffices to construct a subgame perfect equilibrium with outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ with $r_0 = r$, $\sigma_0 = \sigma$, and $p_0 = p$, since we may fix this equilibrium play and augment the strategy profile using backward induction for T periods. To this end, we first claim that there exists $D_0 \in \mathcal{D}$ such that $pD_0(p) = r$ and $\int_p^\infty D_0(v)\,\mathrm{d}v = \sigma$. Indeed, define D_0 as follows:

$$D_0(v) := \begin{cases} 1, & \text{if } v \in \left[0, \frac{p}{p-r}(\mathbb{E}[v]-r-\sigma)\right] \\ \frac{r}{p}, & \text{if } v \in \left(\frac{p}{p-r}(\mathbb{E}[v]-r-\sigma), p+\frac{p}{r}\sigma\right] \\ 0, & \text{if } v > p+\frac{p}{r}\sigma \end{cases}.$$

Since $\sigma \in [(\mathbb{E}[v]-p)^+, S(r/p)-r]$, it follows that $D_0 \in \mathcal{D}$. Moreover, by definition,

$$pD_0(p) = p \cdot \frac{r}{p} = r,$$

and

$$\int_p^\infty D_0(v)\,\mathrm{d}v = \mathbb{E}[v] - \frac{p}{p-r}(\mathbb{E}[v]-r-\sigma) - \frac{r}{p}\left(p - \frac{p}{p-r}(\mathbb{E}[v]-r-\sigma)\right) = \sigma,$$

as desired. Meanwhile, for any $p' \geq 0$ and for any $\omega \geq 0$, consider the following maximization problem:

$$\sup_{D\in\mathcal{D}, q\in[D(p'^+), D(p')]} \left[p'q + \delta\left(\gamma + \beta\int_{p'}^\infty D(v)\,\mathrm{d}v\right)\omega\right]$$
$$\text{s.t. } p'q \leq \underline{r}(\beta)$$

and denote that solution by $(D_{p'}, q_{p'})$ and the value by $\widetilde{\Lambda}(p',\omega)$. Notice that whenever $\underline{r}(\beta) > r^*$, $q_{p'} = D_{p'}(p')$, while $q_{p'} = D_{p'}(p'^+)$ if $\underline{r}(\beta) = r^*$. Moreover, by the definitions of $\overline{\boldsymbol{\omega}}^f(\beta)$ and $\underline{\boldsymbol{\omega}}^f(\beta)$ in the proof of Theorem 3,

$$\widetilde{\Lambda}(p', \overline{\boldsymbol{\omega}}^f(\beta)) \geq \sup_{D\in\mathcal{D}} \left[\alpha p' D(p') + \delta\left(\gamma + \int_{p'}^\infty D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^f(\beta)\right]$$

Lastly, for any $\beta \geq 0$, since

$$\alpha r + \delta(\gamma+\beta\sigma)\overline{\boldsymbol{\omega}}^f(\beta) \geq \sup_{D\in\mathcal{D}} \left[\alpha p D(p) + \delta\left(\gamma + \beta\int_p D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^f(\beta)\right],$$

there exists $\widetilde{\boldsymbol{\omega}}^f(\beta) \leq \overline{\boldsymbol{\omega}}^f(\beta)$ such that $\widetilde{\boldsymbol{\omega}}^f(\beta) < \infty$,

$$\alpha r + \delta(\gamma+\beta\sigma)\widetilde{\boldsymbol{\omega}}^f(\beta) \geq \sup_{D\in\mathcal{D}} \left[\alpha p D(p) + \delta\left(\gamma + \beta\int_p D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^f(\beta)\right], \tag{OA.15}$$

and

$$\widetilde{\Lambda}(p', \widetilde{\boldsymbol{\omega}}^f(\beta)) \geq \sup_{D\in\mathcal{D}} \left[\alpha p' D(p') + \delta\left(\gamma + \int_{p'}^\infty D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^f(\beta)\right]. \tag{OA.16}$$

Now consider the following strategy profile: In period 0, the seller charges price p; the intermediary chooses $D_0 \in \mathcal{D}$ if the seller charges price p, and chooses $D_{p'}$ if the seller charges any other price $p' \neq p$; the tie-breaker chooses $q = D_0(p)$ if the seller charges p and the intermediary chooses D_0, and chooses $q_{p'}$ if the seller charges $p' \neq p$ and the intermediary chooses $D_{p'}$, and always breaks ties in favor of the seller otherwise. Starting from period 1, if the seller charges price p and the intermediary chooses D_0 in period 0, or if the seller charges any $p' \neq p$ and the intermediary chooses $D_{p'}$ in period 0 then they play the finite subgame perfect equilibrium that gives the intermediary payoff $\widetilde{\boldsymbol{\omega}}^f(\beta)$. Otherwise, they play the finite subgame perfect equilibrium that gives the intermediary payoff $\underline{\boldsymbol{\omega}}^f(\beta)$.

We claim that this strategy profile is indeed a finite subgame perfect equilibrium. To see this, notice first that since all players always play a subgame perfect equilibrium from period 1 onward, it suffices to verify that there are no incentives for the seller and the intermediary to deviate from the aforementioned strategies. For the seller, for any $p' \geq 0$,

$$p' q_{p'} \leq \underline{r}(\beta) \leq r = pD_0(p),$$

and hence the seller does not have an incentive to deviate. For the intermediary, given that the seller charges p, and given the continuation play, choosing D_0 gives

$$\alpha r + \delta(\gamma + \beta\sigma)\widetilde{\boldsymbol{\omega}}^f(\beta),$$

whereas the highest payoff the intermediary can obtain from any deviation is

$$\sup_{D\in\mathcal{D}} \left[\alpha p D(p) + \delta\left(\gamma + \beta \int_p D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^f(\beta)\right].$$

By (OA.15), the intermediary has no incentive to deviate when the seller charges p. Finally, if the seller chargers any price $p' \neq p$, following the aforementioned strategy and choosing $D_{p'}$ gives the intermediary payoff

$$\alpha p' q_{p'} + \delta\left(\gamma + \beta\int_{p'}^{\infty} D_{p'}(v)\,\mathrm{d}v\right)\widetilde{\boldsymbol{\omega}}^f(\beta) = \Lambda^\beta(p', \widetilde{\boldsymbol{\omega}})^f(\beta),$$

whereas the highest payoff she can obtain from any deviation is

$$\sup_{D\in\mathcal{D}} \left[\alpha p D(p) + \delta\left(\gamma + \beta \int_p D(v)\,\mathrm{d}v\right)\underline{\boldsymbol{\omega}}^f(\beta)\right].$$

By (OA.16), the intermediary has no incentive to deviate when the seller charges any $p' \neq p$ either. Together, this aforementioned strategy profile is indeed a finite subgame perfect equilibrium.

By construction, this finite subgame perfect equilibrium induces an outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ with $r_0 = r$, $\sigma_0 = \sigma$, and $p_0 = p$, as desired. This completes the proof. ■

OA.2.4 Proof of Proposition 3

Consider any $\beta \in [0, \overline{\beta}]$. We first show that there exists a finite subgame perfect equilibrium that is dominated by any other finite subgame perfect equilibrium. By Corollary 1, it suffices to find a finite subgame perfect equilibrium where the intermediary's normalized continuation payoff is $\underline{\boldsymbol{\omega}}^f(\beta)$ and the normalized sales revenue is $\underline{r}(\beta)$, while the consumer surplus is zero in every period. On the equilibrium path of the finite subgame perfect equilibrium that gives the intermediary payoff $\underline{\boldsymbol{\omega}}^f(\beta)$, which is constructed in the proof of Theorem 3, the seller charges a price $\underline{p}$ such that $\underline{p}\overline{D}(v^{-1}(\underline{p})) = (1-\gamma\delta)\underline{\boldsymbol{\omega}}^f(\beta)/\alpha = \underline{r}(\beta)$; the intermediary chooses the myopic best response when the seller charges $\underline{p}$, which in turn leaves consumers no surplus. As a result, this finite subgame perfect equilibrium induces an outcome $\mathbf{z}^*(\beta)$ that is dominated by any other finite subgame perfect equilibrium outcomes.

Moreover, notice that when $\gamma\delta \leq 1/2$, by its definition, $\underline{\boldsymbol{\omega}}^f(\beta) > \underline{\omega} = \omega^*$ if $\beta < \widehat{\beta}$ is small enough. Therefore, since $\underline{\boldsymbol{\omega}}^f$ is noninreasing on $[0, \overline{\beta}]$, for any γ, δ such that $\gamma\delta \leq 1/2$, there exists $\widehat{\beta}(\gamma, \delta) \in (0, \widehat{\beta})$ such that $\underline{\boldsymbol{\omega}}^f(\beta) > \omega^*$ for all $\beta \in (0, \widehat{\beta}(\gamma, \delta))$. As a result, by its definition, $\underline{\boldsymbol{\omega}}^f$ is strictly decreasing on $(0, \widehat{\beta}(\gamma, \delta))$ and hence for any $\beta, \beta' \in (0, \widehat{\beta}(\gamma, \delta))$ with $\beta' > \beta$, $\mathbf{z}^*(\beta) \succ \mathbf{z}^*(\beta')$. This completes the proof. ■

OA.2.5 Proof of Theorem 4

By Theorem 2, since any Markov perfect equilibrium is a subgame perfect equilibrium, an infinite subgame perfect equilibrium exists if $\beta \geq \widetilde{\beta}$. Conversely, suppose that $\beta < \widetilde{\beta}$. By way of contradiction, suppose that there exists a history at which the intermediary's payoff in some period $T \in \mathbb{N}$ is $\omega_T = \infty$. Then there must exist some $t > T$ such that

$$\delta(\gamma + \beta\sigma_t) \geq 1,$$

where σ_t is the consumer surplus in period t of this history. Rearranging, we have

$$\sigma_t \geq \int_{p^\beta}^{\infty} \overline{D}(v)\,\mathrm{d}v.$$

Meanwhile, since the total surplus in period t of this history must be at most $\mathbb{E}[v]$ and since the sales revenue r_t must be no less than r^* (otherwise the seller can has a profitable deviation in period t of this history). Thus, it must be that

$$\sigma_t \leq \mathbb{E}[v] - r^*.$$

Together, we have

$$\int_{p^\beta}^{\infty} \overline{D}(v)\,\mathrm{d}v \leq \mathbb{E}[v] - r^*,$$

which is equivalent to $\beta \geq \widetilde{\beta}$, a contradiction.

As a result, for any $\beta < \widetilde{\beta}$, $\Omega(\beta) = \Omega^f(\beta)$. Now suppose that $\beta \geq \widetilde{\beta}$. To show that $\Omega(\beta) = [\underline{\omega}, \infty]$, by Theorem 2 and Theorem 3, it suffices to show that for any $\beta \in [\widetilde{\beta}, \widehat{\beta})$ and for any $\hat{\omega} \in [\omega^*, \underline{\boldsymbol{\omega}}^f(\beta)]$, there exists a subgame perfect equilibrium in which the intermediary's payoff is $\hat{\omega}$. To this end, fix any such $\hat{\omega}$. Since $\hat{\omega} \leq \underline{\boldsymbol{\omega}}^f(\beta) \leq \mathbb{E}[v]$, there exists $\hat{p}$ such that $\hat{p}\overline{D}(v^{-1}(\hat{p}))$. As shown in the proof of Theorem 3 (see **Case 2**), $\hat{\omega} \leq \underline{\boldsymbol{\omega}}^f(\beta) \leq \omega^\beta$ implies that $\xi(\hat{p}|\hat{\omega}) = v^{-1}(\hat{p})$. Moreover, for any $p' \geq 0$, let $D_{p'}$ be any solution of $\min_{D \in \mathcal{D}} p' D(p'^{+})$.

Now consider the following strategy profile:

- Start by playing regime $\hat{p}$-MYOPIC. If the seller deviates to $p' \neq \hat{p}$, then enter regime $D_{p'}$-PUNISH immediately. Otherwise, stay in the same regime.
- Under regime $D_{p'}$-PUNISH. If the intermediary deviates, then move to regime $\hat{p}$-MYOPIC. Otherwise, play an infinite Markov perfect equilibrium in the next period.

It remains to show that this strategy profile is a subgame perfect equilibrium. To see this, notice that except for the history at which an infinite Markov perfect equilibrium is played, the intermediary's continuation value is finite. Therefore, by Lemma 1, it suffices to verify that the seller and the intermediary do not have incentives to deviate under each regime. Indeed, under regime $\hat{p}$-MYOPIC, the sales revenue after any deviation of the seller is at most r^*. Therefore, the seller does not have any incentive to deviate. Meanwhile, given price $\hat{p}$, since $\xi(\hat{p}|\hat{\omega}) = v^{-1}(\hat{p})$, the intermediary's best response is indeed myopic. Under regime $D_{p'}$-PUNISH. If the intermediary follows the strategy, then her payoff would be ∞. If she deviates, then her payoff would be at most $\mathbf{W}(p'|\omega^*) < \infty$, and hence the intermediary would not deviate. As a result, the strategy profile above constitutes a subgame perfect equilibrium, and the intermediary's payoff in this equilibrium is $\hat{\omega}$. This completes the proof. ■

OA.2.6 Proof of Corollary 2

For any $\beta < \tilde{\beta}$, by Theorem 4, since every subgame perfect equilibrium is finite, Corollary 1 ensures any subgame perfect equilibrium outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ must be such that $(r_t, \sigma_t, p_t) \in \mathbf{Z}^f(\beta) = \mathbf{Z}(\beta)$. Moreover, for any $(r, \sigma, p) \in \mathbf{Z}(\beta) = \mathbf{Z}^f(\beta)$ and for any $T \geq 0$, Corollary 1 ensures that there exists a subgame perfect equilibrium outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ such that $r_T = r$, $\sigma_T = \sigma$, and $p_T = p$.

Now consider any $\beta \geq \widetilde{\beta}$. First consider any subgame perfect equilibrium outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$. For any $t \geq 0$, clearly $r^* \leq r_t \leq p_t$. Furthermore, let D_t be the disclosure policy chosen by the intermediary on the equilibrium path in period t, it then follow that

$$\sigma_t = \int_{p_t}^{\infty} D_t(v)\,\mathrm{d}v.$$

Since $D_t \in \mathcal{D}$, $\sigma_t \geq (\mathbb{E}[v] - p_t)^+$. Moreover, since $D_t \in \mathcal{D}$ is nonincreasing, it must be that

$$\int_{p_t}^{\infty} \overline{D}(v)\,\mathrm{d}v \geq \int_{p_t}^{\infty} D_t(v)\,\mathrm{d}v \geq \sigma_t - (p - p_t)D_t(p_t),$$

for all $p \geq 0$. As a result,

$$\sigma_t \leq \min_{p \geq 0} \left[\int_p^{\infty} \overline{D}(v) + (p - p_t)D_t(p_t) \right] = S\left(\frac{r_t}{p_r}\right) - r_t,$$

where the equality follows from the first order condition of the minimization problem, which implies that at the solution $\hat{p}_t$, $\overline{D}(\hat{p}_t) = D_t(p_t) = r_t/p_t$. Together, we have

$$(\mathbb{E}[v] - p_t)^+ \leq \sigma_t \leq S\left(\frac{r_t}{p_r}\right) - r_t,$$

for all $t \geq 0$.

Conversely, consider any $(r, \sigma, p) \in \mathbf{Z}(\beta)$ and any $T \geq 0$. As in the proof of Corollary 1, it suffices to find a subgame perfect equilibrium with outcome $\mathbf{z} = \{r_t, \sigma_t, \omega_t, p_t, m_t\}$ such that $r_0 = r$, $\sigma_0 = \sigma$, and $p_0 = p$. As shown in the proof of Corollary 1, since $(\mathbb{E}[v] - p)^+ \leq \sigma \leq S(r/p) - r$, there exists $D_0 \in \mathcal{D}$ such that $pD_0(p) = r_0$ and $\int_p^{\infty} D_0(v)\,\mathrm{d}v = \sigma$.

Now consider the following strategy profile: In period 0, the seller charges price p; the intermediary chooses $D_0 \in \mathcal{D}$ if the seller charges p, and chooses any solution of $\min_{D \in \mathcal{D}} p'D('^+)$ if the seller charges $p' \neq p$; and the tie-breaker breaks tie in favor of the seller when the seller charges p, and against the seller when he charges $p' \neq p$. From period 1 onward, if the seller charges p in period 0 and if the intermediary chooses D_0, or if the seller charges any $p' \neq p$ and the intermediary chooses a solution of $\min_{D \in \mathcal{D}} p'D('^+)$, then all players play an infinite Markov perfect equilibrium. Otherwise, the play the subgame perfect equilibrium that gives the intermediary equilibrium payoff ω^*.

We claim that this strategy profile constitutes a subgame perfect equilibtium. To see this, first notice that it suffices to verify that both the seller and the intermediary do not have any incentive to deviate in period 0. For the seller, given the intermediary's strategy, charging price p gives payoff $(1 - \alpha)r$, while the largest possible revenue from deviation is $(1 - \alpha)r^* \leq (1 - \alpha)r$. Thus, the seller does not have any incentive to deviate. For the intermediary, if she follows the strategy, then the continuation play is an infinite Markov perfect equilibrium and hence her payoff would be ∞, whereas if she deviates after seeing any price $p' \geq 0$, her normalized continuation payoff would be $\omega^* < \infty$. Therefore, the intermediary does not have any incentive to deviate either. This completes the proof. ■

OA.2.7 Proof of Proposition 4

This proposition immediately follows from Proposition 3 and Theorem 4 after letting $\widetilde{\beta}(\gamma,\delta) := \min\{\widetilde{\beta}, \widehat{\beta}(\gamma,\delta)\}$. ■

OA.3 Omitted Proofs for Section 6

OA.3.1 Proof of Proposition 5

Consider the strategy profile where the seller charges $\overline{p}$, the intermediary chooses $\overline{D}$ after observing any prices, and the tie breaker breaks ties in favor of the seller regardless in every period. Clearly this strategy profile is Markov. Moreover, given the seller's strategy at any history, choosing $\overline{D}$ is always a best response for the intermediary since $\overline{D}$ maximizes the subscription fee and the market growth rate at the same time. Lastly, given that the intermediary always chooses $\overline{D}$, $\overline{p}$ is the unique best response for the seller at any history. Therefore, this strategy profile is indeed a Markov perfect equilibrium.

Furthermore, when $\beta < \overline{\beta}$, we have

$$\delta\left(\gamma + \beta \int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v\right) < 1$$

and hence the intermediary's payoff in the Markov perfect equilibrium described above is

$$\rho^{\mathrm{M}} = \frac{\widetilde{\alpha}\int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v}{1-\delta\left(\gamma+\beta\int_{\overline{p}}^{\infty}\overline{D}(v)\,\mathrm{d}v\right)} < \infty.$$

Therefore, this Markov perfect equilibrium is finite.

To see that this finite Markov perfect equilibrium induces a unique outcome whenever $\beta < \overline{\beta}$, consider any other finite Markov perfect equilibrium outcome $\mathbf{y} = (r, \sigma, \omega, p, \{m_t\})$. Since $\omega < \infty$, by the same arguments as the proof of, Lemma 1, it follows that the intermediary must choose $D \in \mathcal{D}$ that attains

$$\omega = \sup_{D\in\mathcal{D}}\left[\widetilde{\alpha}\int_{p}^{\infty} D(v)\,\mathrm{d}v + \delta\left(\gamma+\beta\int_{p}^{\infty} D(v)\,\mathrm{d}v\right)\omega\right].$$

As a result, it follows that $\omega = \rho^{\mathrm{M}}$, $\sigma = \int_{\overline{p}}^{\infty}\overline{D}(v)\,\mathrm{d}v$, $p = \overline{p}$, and $r = \overline{p}\overline{D}(\overline{p})$.

Lastly, for any β, β' such that $0 < \beta < \beta' < \overline{\beta}$, the intermediary's equilibrium payoffs is higher under β', and the market growth rate is also higher under β'. Thus, it must be that $\mathbf{y}^{\mathrm{M}}(\beta) \prec \mathbf{y}^{\mathrm{M}}(\beta')$. This completes the proof. ■

OA.3.2 Proof of Proposition 6

Notice that by Proposition 1 and Proposition 5, ω^{M} is nonincreasing on $[0,\overline{\beta})$ and ρ^{M} is nondecreasing on $[0,\overline{\beta})$. Moreover, if

$$\frac{\widetilde{\alpha}}{\alpha} < \frac{\mathbb{E}[v]}{\int_{\overline{p}}^{\infty}\overline{D}(v)\,\mathrm{d}v},$$

then

$$\rho^{\mathrm{M}}(0) = \frac{\widetilde{\alpha}\int_{\overline{p}}^{\infty}\overline{D}(v)\,\mathrm{d}v}{1-\gamma\delta} < \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta} = \omega^{\mathrm{M}}(0).$$

Thus, there exists $\beta^0 > 0$ such that $\rho^{\mathrm{M}}(\beta) < \omega^{\mathrm{M}}(\beta)$ for all $\beta < \beta^0$.

In the meantime, if

$$\frac{\widetilde{\alpha}}{\alpha} + 1 < \frac{\mathbb{E}[v]}{\int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v},$$

then

$$\rho^{\mathrm{M}}(\underline{\beta}) = \frac{\widetilde{\alpha}\int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v}{1-\delta\left(\gamma + \frac{1-\gamma\delta}{\delta\mathbb{E}[v]}\int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)} < \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta} = \omega^{\mathrm{M}}(\underline{\beta}).$$

Hence, there exists $\beta^0 > \underline{\beta}$ such that $\rho^{\mathrm{M}}(\beta) < \omega^{\mathrm{M}}(\beta)$ for all $\beta < \beta^0$. This completes the proof. ■

OA.3.3 Omitted Proofs for Section 6.2

First, observe that when $\chi < 1/\delta$, the intermediary's payoff must be finite in any Markov perfect equilibrium since the the market size can grow at most at a rate χ. We focus on caps less than $1/\delta$, in which case only finite Markov perfect equilibria can exist. We characterize the equilibria in this case and then use the characterization to prove Propositions 7, 8, 9, and 10.

By similar arguments as those in the baseline model, the one-shot deviation principle holds in this setting, and an analog of Lemma 2 can be established as in Lemma Lemma OA.4.

Lemma OA.4. *A finite Markov perfect equilibrium is characterized by a tuple $\{\omega^{\mathrm{M}}, p^{\mathrm{M}}, \mathbf{D}^{\mathrm{M}}\}$ with $\omega^{\mathrm{M}} \geq 0$, $p^{\mathrm{M}} \geq 0$, and $\mathbf{D}^{\mathrm{M}} : \mathbb{R}_+ \to \mathcal{D}$ that satisfy the following conditions:*

$$\omega^{\mathrm{M}} = \sup_{D\in\mathcal{D}} \left\{ \alpha p^{\mathrm{M}} D(p^{\mathrm{M}}) + \delta\omega^{\mathrm{M}} \min\left[\chi, \left(\gamma + \beta\int_{p^{\mathrm{M}}}^{\infty} D(v)\,\mathrm{d}v\right)\right]\right\}, \tag{OA.17}$$

$$p^{\mathrm{M}}\mathbf{D}_t^{\mathrm{M}}(p^{\mathrm{M}}|p^{\mathrm{M}}) \geq p\mathbf{D}^{\mathrm{M}}(p|p), \tag{OA.18}$$

for all $p \geq 0$, and

$$\alpha p\mathbf{D}^{\mathrm{M}}(p|p) + \delta\omega^{\mathrm{M}} \min\left\{\chi, \left(\gamma + \beta\int_{p}^{\infty} \mathbf{D}^{\mathrm{M}}(v|p)\,\mathrm{d}v\right)\right\} \geq \alpha pD(p) + \delta\omega^{\mathrm{M}} \min\left\{\chi, \left(\gamma + \beta\int_{p}^{\infty} D(v)\,\mathrm{d}v\right)\right\}, \tag{OA.19}$$

for all $p \geq 0$ and for all $D \in \mathcal{D}$.

Lemma OA.5. *For any $\alpha, \delta \in (0,1), \beta, \omega \geq 0$ and for any $p \geq 0$,*

$$\Delta(p|\beta,\chi) := \operatorname*{argmax}_{D\in\mathcal{D}} \left[\alpha pD(p) + \delta\omega \min\left\{\chi, \left(\gamma + \beta\int_{p}^{\infty} D(v)\,\mathrm{d}v\right)\right\}\right]$$

is nonempty. Moreover, for any $D \in \Delta(p|\beta,\chi)$,

$$D(v) = \overline{D}(\xi(p|\beta,\chi)), \tag{OA.20}$$

for all $v \in [\xi(p|\beta,\chi), p]$ and

$$\int_{\xi(p|\beta,\chi)}^{\infty} D(v)\,\mathrm{d}v = \int_{\xi(p|\beta,\chi)}^{\infty} \overline{D}(v)\,\mathrm{d}v, \tag{OA.21}$$

where

$$\xi(p|\beta,\chi) := \min\left\{\bar{\xi}(p|\beta,\chi), \max\left\{\left(1-\frac{\alpha}{\delta\beta\omega}\right)^{+} p, v^{-1}(p)\right\}\right\}.$$

and $\bar{\xi}(p|\beta,\chi)$ *is such that*

$$\begin{aligned} \bar{\xi}(p|\beta,\chi) &= p, && \text{if } \chi > \gamma + \beta \int_p^\infty \overline{D}(v)\,\mathrm{d}v \\ \bar{\xi}(p|\beta,\chi) &= v^{-1}(p), && \text{if } \chi < \gamma + \beta \int_p^\infty \underline{D}(v)\,\mathrm{d}v \\ \gamma + \beta\left(\textstyle\int_{\bar{\xi}(p|\beta,\chi)}^\infty \overline{D}(v)\,\mathrm{d}v - (p-\bar{\xi}(p|\beta,\chi))\overline{D}(\bar{\xi}(p|\beta,\chi))\right) &= \chi, && \text{otherwise} \end{aligned}$$

Proof. Arguments similar to the proof of Lemma 2 imply that the intermediary's problem can be simplified to

$$\max_{\xi\in[v^{-1}(p),p]} \alpha p\overline{D}(\xi) + \delta\omega \max\left\{\chi, \gamma + \beta\left(\int_\xi^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi)\right)\right\}, \tag{OA.22}$$

By definition of $\bar{\xi}(p|\beta,\chi)$, the intermediary's objective function in (OA.22))equals to

$$\begin{cases} \alpha p\overline{D}(\xi) + \delta\omega\left(\gamma + \beta\left(\int_\xi^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi)\right)\right), & \text{if } \xi \in [v^{-1}(p), \bar{\xi}(p|\beta,\chi)] \\ \alpha p\overline{D}(\xi) + \delta\omega\chi, & \text{if } \xi \in (\bar{\xi}(p|\beta,\chi), p] \end{cases}$$

From Lemma 2, we know that $\xi = \max\{(1-\frac{\alpha}{\delta\beta\omega})^+ p, v^{-1}(p)\}$ maximizes

$$\alpha p\overline{D}(\xi) + \delta\omega\left(\gamma + \beta\left(\int_\xi^\infty \overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi)\right)\right)$$

on the interval $[v^{-1}(p), p]$. Notice that $\alpha p\overline{D}(\xi) + \delta\omega\chi$ is decreasing in ξ. Hence, the optimal solution to (OA.22) is either $\bar{\xi}(p|\beta,\chi)$ or $\max\{(1-\frac{\alpha}{\delta\beta\omega})^+ p, v^{-1}(p)\}$, whichever is smaller. That is,

$$\xi(p|\beta,\chi) := \min\left\{\bar{\xi}(p|\beta,\chi), \max\left\{\left(1-\frac{\alpha}{\delta\beta\omega}\right)^{+} p, v^{-1}(p)\right\}\right\}.$$

■

Given any $\alpha \in [0,1], \beta \geq 0, \delta \in (0,1), \omega \geq 0$, let $\hat{p}(\omega)$ denote the optimal price when there is no cap on the market growth rate:

$$\hat{p}(\omega) := \text{argmax}\left\{\min\left\{p\overline{D}\left(\left(1-\frac{\alpha}{\delta\beta\omega}\right)^{+} p\right), p\overline{D}(v^{-1}(p))\right\}\right\}$$

Let $\hat{r}(\omega)$ denote the corresponding sales revenue under $\hat{p}(\omega)$:

$$\hat{r}(\omega) := \min\left\{\hat{p}(\omega)\left(\left(1-\frac{\alpha}{\delta\beta\omega}\right)^{+}\hat{p}(\omega)\right), \hat{p}(\omega)\overline{D}(v^{-1}(\hat{p}(\omega)))\right\}.$$

Lemma OA.6. *For any* $\chi \in (\gamma, 1/\delta)$, $\alpha \in [0,1], \beta \geq 0, \delta \in (0,1), \omega \geq 0$ *and for any selection* $\mathbf{D}$ *of* $\Delta(\cdot|\beta,\chi)$, *the maximization problem*

$$\max_{p\geq 0} p\mathbf{D}(p|p)$$

has a non-empty set of solutions $\tilde{P}$ *such that*

$$\tilde{P} = \begin{cases} \{\hat{p}(\omega)\}, & \text{if } \hat{r}(\omega) > \mathbb{E}(v) - \frac{\chi-\gamma}{\beta} \\ \{\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}\}, & \text{if } \hat{r}(\omega) < \mathbb{E}(v) - \frac{\chi-\gamma}{\beta} \\ \{\hat{p}(\omega), \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}\}, & \text{if } \hat{r}(\omega) = \mathbb{E}(v) - \frac{\chi-\gamma}{\beta} \end{cases}.$$

Furthermore,

$$v^{-1}(\hat{p}(\omega)) \le \left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ \hat{p}(\omega) \le \overline{p}$$

with at least one inequality binding. In addition,

$$\tilde{p} \in \tilde{P} \text{ and } \int_{\tilde{p}}^{\infty} \mathbf{D}(v|\tilde{p})\,\mathrm{d}v = 0 \iff v^{-1}(\tilde{p}) = \left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ \tilde{p} \le \overline{p}.$$

Proof. By Lemma OA.5, for any selection $\mathbf{D}$ of $\Delta(\cdot|\beta,\chi)$,

$$\begin{aligned} p\mathbf{D}(p|p) = p\overline{D}(\xi(p|\beta,\chi)) &= p\overline{D}\left(\min\left\{\overline{\xi}(p|\beta,\chi), \max\left\{\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ p, v^{-1}(p)\right\}\right\}\right) \\ &= \max\left\{p\overline{D}(\overline{\xi}(p|\beta,\chi)), \min\left\{p\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ p\right), p\overline{D}(v^{-1}(p))\right\}\right\}, \end{aligned}$$

where the last equality follows from the fact that $\overline{D}$ is strictly decreasing. We will separately analyze the two terms in the maximization operator: $p\overline{D}(\overline{\xi}(p|\beta,\chi))$ and $\min\{p\overline{D}((1-\frac{\alpha}{\delta\beta\omega})^+p), p\overline{D}(v^{-1}(p))\}$, and then draw a conclusion based on the analysis.

From the proof of Lemma 4, we know that the second term $\min\{p\overline{D}((1-\frac{\alpha}{\delta\beta\omega})^+p), p\overline{D}(v^{-1}(p))\}$ is quasi-concave and has a unique maximizer $\hat{p}(\omega)$ such that $v^{-1}(\hat{p}(\omega)) \le (1-\frac{\alpha}{\delta\beta\omega})^+\hat{p}(\omega) \le \overline{p}$, with at least one inequality binding.

We now study the properties of the first term $p\overline{D}(\overline{\xi}(p|\beta,\chi))$ by considering three cases. First, for any price $p \le \mathbb{E}(v) - (\chi-\gamma)/\beta$, we have $\chi < \gamma + \beta\int_p^\infty \underline{D}(v)\,\mathrm{d}v$. Thus, $\overline{\xi}(p|\beta,\chi) = 0$ and $p\overline{D}(\overline{\xi}(p|\beta,\chi)) = p$, which is increasing in p and always weakly larger than $\min\{p\overline{D}((1-\frac{\alpha}{\delta\beta\omega})^+p), p\overline{D}(v^{-1}(p))\}$. Second, for any price p such that $p > \mathbb{E}(v) - (\chi-\gamma)/\beta$ and $\chi < \gamma + \beta\int_p^\infty \overline{D}(v)\,\mathrm{d}v$, we have, by definition of $\overline{\xi}(p|\beta,\chi)$,

$$\gamma + \beta\left(\int_{\overline{\xi}(p|\beta,\chi)}^{\infty} \overline{D}(v)\,\mathrm{d}v - (p - \overline{\xi}(p|\beta,\chi))\overline{D}(\overline{\xi}(p|\beta,\chi))\right) = \chi$$

which implies

$$p\overline{D}(\overline{\xi}(p|\beta,\chi)) = \int_{\overline{\xi}(p|\beta,\chi)}^{\infty} \overline{D}(v)\,\mathrm{d}v + \overline{\xi}(p|\beta,\chi)\overline{D}(\overline{\xi}(p|\beta,\chi)) - \frac{\chi-\gamma}{\beta}.$$

This in turn implies that $p\overline{D}(\overline{\xi}(p|\beta,\chi))$ is decreasing in p, because $\overline{\xi}(p|\beta,\chi)$ is increasing in p and $\int_{\overline{\xi}(p|\beta,\chi)}^\infty \overline{D}(v)\,\mathrm{d}v + \overline{\xi}(p|\beta,\chi)\overline{D}(\overline{\xi}(p|\beta,\chi))$ is decreasing in $\overline{\xi}(p|\beta,\chi))$. Lastly, for any price even larger such that it satisfies $\chi \ge \gamma + \beta\int_p^\infty \overline{D}(v)\,\mathrm{d}v$, we have $\overline{\xi}(p|\beta,\chi) = p$ and $p\overline{D}(\overline{\xi}(p|\beta,\chi)) = p\overline{D}(p)$, in which case we always have

$$p\overline{D}(\overline{\xi}(p|\beta,\chi)) = p\overline{D}(p) < \min\left\{p\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ p\right), p\overline{D}(v^{-1}(p))\right\},$$

which means that the seller's objective function is equal to the second term in the maximization operator.

Combining the above properties of $p\overline{D}(\overline{\xi}(p|\beta,\chi))$ with the fact that $\min\{p\overline{D}((1-\frac{\alpha}{\delta\beta\omega})^{+}p), p\overline{D}(v^{-1}(p))\}$ is quasi-concave, we know that the seller's optimal price $\tilde{p}$ must be either $\hat{p}(\omega)$ or $\tilde{p}_2 := \mathbb{E}(v) - (\chi-\gamma)/\beta$, whichever generates a higher sales revenue.

Finally, by definition of $\overline{\xi}(p|\beta,\chi)$, we have that $\overline{\xi}(\tilde{p}_2|\beta,\chi) = 0$. Thus, $\int_{\tilde{p}_2}^{\infty} \mathbf{D}(v|\tilde{p}_2)\,\mathrm{d}v = \frac{\chi-\gamma}{\beta} > 0$. So, $\int_{\tilde{p}}^{\infty} \mathbf{D}(v|\tilde{p})\,\mathrm{d}v = 0$ can only occur when $\tilde{p} = \hat{p}(\omega)$. According to Lemma 4, we therefore have

$$\int_{\tilde{p}}^{\infty} \mathbf{D}(v|\tilde{p})\,\mathrm{d}v = 0 \iff v^{-1}(\tilde{p}) = \left(1 - \frac{\alpha}{\delta\beta\omega}\right)^{+} \tilde{p} \leq \overline{p}.$$

■

Lemma OA.7. *If $p^{\mathrm{M}} \neq \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$, then $\mathbf{z}^{\mathrm{M}} = (r^{\mathrm{M}}, \sigma^{\mathrm{M}}, \omega^{\mathrm{M}}, p^{\mathrm{M}}, \{m_t^{\mathrm{M}}\})$ is a finite Markov perfect equilibrium outcome if and only if $\mathbf{z}^{\mathrm{M}}$ satisfies characterization in Theorem 1 and one of the following conditions holds:*

1. *$\beta \leq \underline{\beta}$,*
2. *$\beta \in (\underline{\beta}, \overline{\beta})$ and $\chi \geq \beta\left(\mathbb{E}(v) - \frac{1-\gamma\delta}{\alpha} g^{\beta}(p^{\beta})\right) + \gamma$,*
3. *$\beta = \overline{\beta}$ and*

$$\omega^{\mathrm{M}} \in \left[g^{\beta}(\overline{p}), \min\left\{\frac{\alpha}{\delta\beta}\left(1 - \frac{\overline{p}\overline{D}(\overline{p})}{\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}}\right)^{-1}, \frac{\alpha}{\delta\beta}\left(1 + \frac{\overline{p}\overline{D}(\overline{p})}{\int_{\overline{p}} \overline{D}(v)\,\mathrm{d}v - \frac{\chi-\gamma}{\beta}}\right)\right\}\right].$$

Proof. By Lemma OA.5 and Lemma OA.6, if $p^{\mathrm{M}} \neq \mathbb{E}(v) - (\chi-\gamma)/\beta$, it must be that p^{M} maximizes

$$p\overline{D}\left(\max\left\{\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^{+} p, v^{-1}(p)\right\}\right)$$

and the cap on market growth rate is not binding. By Lemma OA.4 and Lemma OA.6, such an equilibrium outcome must also be an equilibrium outcome in the baseline model where there is not a cap on the market growth rate (i.e., the baseline model). Thus, it suffices to focus on the equilibrium outcomes characterized in Theorem 1 and examine which of them remain an equilibrium outcome when there is a cap on the market growth rate.

Case 1: $\beta \in [0, \underline{\beta}]$.

Consider any equilibrium outcome $\mathbf{z}^{\mathrm{M}}$ in characterized in Theorem 1 for the case when $\beta \leq \underline{\beta}$. We have $p^{\mathrm{M}} = \mathbb{E}(v)$ and $r^{\mathrm{M}} = \mathbb{E}(v)$. Hence, $\hat{p}(\omega^{\mathrm{M}}) = \hat{r}(\omega^{\mathrm{M}}) = \mathbb{E}(v)$. According to Lemma OA.6, since $\hat{r}(\omega^{\mathrm{M}}) = \mathbb{E}(v) > \mathbb{E}(v) - (\chi-\gamma)/\delta$, the seller's optimal price(s) is $\tilde{P} = \{\hat{p}(\omega^{\mathrm{M}})\}$, so setting a price at $\mathbb{E}(v)$ is optimal for the seller. Also, according to Theorem 1, the consumer surplus is $\sigma^{\mathrm{M}} = 0$ and the market growth rate is γ. Since the market growth rate γ is below the cap χ on the market growth rate, we have

$$\begin{aligned}\sup_{D\in\mathcal{D}}\left\{\alpha p^{\mathrm{M}}D(p^{\mathrm{M}})+\delta\omega^{\mathrm{M}}\min\left[\chi,\left(\gamma+\beta\int_{p^{\mathrm{M}}}^{\infty}D(v)\,\mathrm{d}v\right)\right]\right\}=&\alpha p^{\mathrm{M}}\mathbf{D}^{\mathrm{M}}(p^{\mathrm{M}})+\delta\omega^{\mathrm{M}}\left(\gamma+\beta\int_{p^{\mathrm{M}}}^{\infty}\mathbf{D}^{\mathrm{M}}(v)\,\mathrm{d}v\right)\\=&\alpha\mathbb{E}[v]+\gamma\delta\omega^{\mathrm{M}}\\=&\omega^{\mathrm{M}}\\=&\frac{\alpha p^{\mathrm{M}}\mathbf{D}^{\mathrm{M}}(p^{\mathrm{M}}|p^{\mathrm{M}})}{1-\delta\left(\gamma+\beta\int_{p^{\mathrm{M}}}^{\infty}\mathbf{D}^{\mathrm{M}}(v|p)\,\mathrm{d}v\right)}\end{aligned}$$

Therefore, by Lemma OA.4, $\mathbf{z}^{\mathrm{M}}$ is an equilibrium outcome.

Case 2: $\beta\in(\underline{\beta},\overline{\beta})$.

In this case, since the equilibrium outcome satisfies the conditions in Theorem 1, we have $p^{\mathrm{M}}=p^{\beta}$, $r^{\mathrm{M}}=\frac{1-\gamma\delta}{\alpha}g^{\beta}(p^{\beta})$ and $\sigma^{\mathrm{M}}=0$. According to Lemma OA.6, this is an equilibrium if it is optimal for the seller to price at $p^{\mathrm{M}}=p^{\beta}$ rather than $p=\mathbb{E}(v)-(\chi-\gamma)/\beta$, i.e.,

$$\frac{1-\gamma\delta}{\alpha}g^{\beta}(p^{\beta})\geq\mathbb{E}(v)-\frac{\chi-\gamma}{\beta}.$$

Equivalently,

$$\chi\geq\beta\left(\mathbb{E}(v)-\frac{1-\gamma\delta}{\alpha}g^{\beta}(p^{\beta})\right)+\gamma.$$

Case 3: $\beta=\overline{\beta}$.

In this case, if the equilibrium outcome satisfies the conditions in Theorem 1, we have $\omega^{\mathrm{M}}\geq g^{\beta}(\overline{p})$, $p^{\mathrm{M}}=\frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}}-\alpha}\overline{p}$, $r^{\mathrm{M}}=\frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}}-\alpha}\overline{p}\overline{D}(\overline{p})$, $\sigma^{\mathrm{M}}=\int_{\overline{p}}^{\infty}\overline{D}(v)\,\mathrm{d}v-\frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}}-\alpha}\overline{p}\overline{D}(\overline{p})$. For any of these equilibrium outcomes to be an equilibrium when there is a cap χ on the market growth rate, we need the seller to find it optimal to choose p^{M} rather than $p=\mathbb{E}(v)-(\chi-\gamma)/\beta$, and the equilibrium growth rate does not exceed χ.

For the seller to find it optimal to choose p^{M}, we need

$$\mathbb{E}(v)-\frac{\chi-\gamma}{\beta}\leq\frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}}-\alpha}\overline{p}\overline{D}(\overline{p})$$

which is equivalent to

$$\omega^{\mathrm{M}}\leq\frac{\alpha}{\delta\beta}\left(1-\frac{\overline{p}\overline{D}(\overline{p})}{\mathbb{E}(v)-\frac{\chi-\gamma}{\beta}}\right)^{-1}.$$

For the equilibrium growth rate to not exceed χ, we need

$$\gamma+\beta\sigma^{\mathrm{M}}\leq\chi$$

which is equivalent to

$$\omega^{\mathrm{M}}\leq\frac{\alpha}{\delta\beta}\left(1+\frac{\overline{p}\overline{D}(\overline{p})}{\int_{\overline{p}}^{\infty}\overline{D}(v)\,\mathrm{d}v-\frac{\chi-\gamma}{\beta}}\right).$$

Thus, there is a continuum of equilibrium with non-binding cap with the intermediary's equilibrium normalized payoff equal to

$$\omega^{\mathrm{M}}\in\left[g^{\beta}(\overline{p}),\min\left\{\frac{\alpha}{\delta\beta}\left(1-\frac{\overline{p}\overline{D}(\overline{p})}{\mathbb{E}(v)-\frac{\chi-\gamma}{\beta}}\right)^{-1},\frac{\alpha}{\delta\beta}\left(1+\frac{\overline{p}\overline{D}(\overline{p})}{\int_{\overline{p}}\overline{D}(v)\,\mathrm{d}v-\frac{\chi-\gamma}{\beta}}\right)\right\}\right]$$

■

We define

$$\ddot{\beta} := \frac{1-\gamma\delta}{\delta}\frac{1}{\mathbb{E}(v) - \overline{p}\overline{D}(\overline{p})}.$$

Note that $\ddot{\beta} > \underline{\beta}$. In addition, $\ddot{\beta} < \overline{\beta}$ is equivalent to $\mathbb{E}(v) > \overline{p}\overline{D}(\overline{p}) + \int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v$, which always holds because $\overline{p}\overline{D}(\overline{p}) + \int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v < \int_0^{\overline{p}} \overline{D}(v)\,\mathrm{d}v + \int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v = \mathbb{E}(v)$. Thus. $\ddot{\beta} \in (\underline{\beta}, \overline{\beta})$.

Lemma OA.8. *If $p^{\mathrm{M}} = \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$, then $\mathbf{z}^{\mathrm{M}} = (r^{\mathrm{M}}, \sigma^{\mathrm{M}}, \omega^{\mathrm{M}}, p^{\mathrm{M}}, \{m_t^{\mathrm{M}}\})$ is a finite Markov perfect equilibrium outcome if and only if $r^{\mathrm{M}} = \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$, $\sigma^{\mathrm{M}} = \frac{\chi-\gamma}{\beta}$, $\omega^{\mathrm{M}} = \frac{\alpha}{1-\delta\chi}\left(\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}\right)$, $m_t^{\mathrm{M}} = \chi^t$, and one of the following conditions holds:*

1. *$\beta \geq \underline{\beta}$, $\omega^{\mathrm{M}} \leq g^{\beta}(\overline{p})$, and $\mathbb{E}(v) - \frac{\chi-\gamma}{\beta} \geq \frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}}-\alpha}(g^{\beta})^{-1}(\omega^{\mathrm{M}})\overline{D}((g^{\beta})^{-1}(\omega^{\mathrm{M}}))$,*
2. *$\beta \geq \ddot{\beta}$ and $\omega^{\mathrm{M}} > g^{\beta}(\overline{p})$.*

Proof. Since $p^{\mathrm{M}} = \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$, by Lemma OA.5 and Lemma OA.6, we must have $r^{\mathrm{M}} = \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$, $\sigma^{\mathrm{M}} = \frac{\chi-\gamma}{\beta}$. So, the market size grows at a rate of χ, i.e., $m_t^{\mathrm{M}} = \chi^t$. The equilibrium payoff of the intermediary (per unit of the current market size) is therefore

$$\omega^{\mathrm{M}} = \frac{\alpha}{1-\chi\delta}\left(\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}\right).$$

For $\mathbf{z}^{\mathrm{M}}$ to be an equilibrium outcome, the seller must find it optimal to price at $p^{\mathrm{M}} = \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$ rather than $\hat{p}(\omega)$, i.e.,

$$\mathbb{E}(v) - \frac{\chi-\gamma}{\beta} \geq \max_p \left\{\min\left[p\overline{D}\left(\left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{M}}}\right)^+ p\right), p\overline{D}\left(v^{-1}(p)\right)\right]\right\}.$$

We consider two cases according to whether or not $\omega^{\mathrm{M}} > \alpha/\delta\beta$, i.e.,

$$\omega^{\mathrm{M}} = \frac{\alpha}{1-\delta\chi}\left(\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}\right) > \frac{\alpha}{\delta\beta} \quad \Leftrightarrow \quad \beta > \underline{\beta}$$

Case 1: $\beta < \underline{\beta}$.
In this case, the seller's revenue when the price is $\hat{p}(\omega^{\mathrm{M}})$ is

$$\max_p \left\{\min\left[p\overline{D}\left(\left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{M}}}\right)^+ p\right), p\overline{D}\left(v^{-1}(p)\right)\right]\right\} = \max_p p\overline{D}\left(v^{-1}(p)\right) = \mathbb{E}(v)$$

Hence, the seller is able to obtain a revenue of $\mathbb{E}(v)$ by charging a price at $p = \mathbb{E}(v)$ which is always higher than the revenue he could get by charging a price $p = \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$ and induce a binding cap on the market growth rate. So, there is no equiliria in which the cap of market growth is binding when $\beta < \underline{\beta}$.

Case 2: $\beta \geq \underline{\beta}$.
In this case, we either have $v^{-1}(\hat{p}(\omega^{\mathrm{M}})) = \left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{M}}}\right)\hat{p}(\omega^{\mathrm{M}}) \leq \overline{p}$, or $v^{-1}(\hat{p}(\omega^{\mathrm{M}})) < \left(1-\frac{\alpha}{\delta\beta\omega^{\mathrm{M}}}\right)\hat{p}(\omega^{\mathrm{M}}) = \overline{p}$. We consider the two cases one by one.

Sub-case 1: $v^{-1}(\hat{p}(\omega^{\mathrm{M}})) = \left(1 - \frac{\alpha}{\delta\beta\omega^{\mathrm{M}}}\right)\hat{p}(\omega^{\mathrm{M}}) \leq \overline{p}$.
Notice that $v^{-1}(\hat{p}(\omega^{\mathrm{M}})) = \left(1 - \frac{\alpha}{\delta\beta\omega^{\mathrm{M}}}\right)\hat{p}(\omega^{\mathrm{M}}) \leq \overline{p}$ is equivalent to

$$\omega^{\mathrm{M}} = g^{\beta}\left(\left(1 - \frac{\alpha}{\delta\beta\omega^{\mathrm{M}}}\right)\hat{p}(\omega^{\mathrm{M}})\right) \leq g^{\beta}(\overline{p}).$$

Therefore,

$$\hat{p}(\omega^{\mathrm{M}}) = \frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}} - \alpha}(g^{\beta})^{-1}(\omega^{\mathrm{M}}).$$

The seller finds it optimal to choose $p = \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$, rather than $\hat{p}(\omega^{\mathrm{M}})$, if

$$\mathbb{E}(v) - \frac{\chi - \gamma}{\beta} \geq \frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}} - \alpha}(g^{\beta})^{-1}(\omega^{\mathrm{M}})\overline{D}((g^{\beta})^{-1}(\omega^{\mathrm{M}})).$$

Thus, by Lemma OA.6, $\mathbf{z}^{\mathrm{M}}$ is an equilibrium in this sub-case if $\omega^{\mathrm{M}} = \frac{\alpha}{1-\delta\chi}\left(\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}\right) \leq g^{\beta}(\overline{p})$ and $\mathbb{E}(v) - \frac{\chi-\gamma}{\beta} \geq \frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}}-\alpha}(g^{\beta})^{-1}(\omega^{\mathrm{M}})\overline{D}((g^{\beta})^{-1}(\omega^{\mathrm{M}}))$.

Sub-case 2: $v^{-1}(\hat{p}(\omega^{\mathrm{M}})) < \left(1 - \frac{\alpha}{\delta\beta\omega^{\mathrm{M}}}\right)\hat{p}(\omega^{\mathrm{M}}) = \overline{p}$.
In this case, we have

$$\omega^{\mathrm{M}} > g^{\beta}\left(\left(1 - \frac{\alpha}{\delta\beta\omega^{\mathrm{M}}}\right)\hat{p}(\omega^{\mathrm{M}})\right) = g^{\beta}(\overline{p}).$$

and

$$\hat{p}(\omega^{\mathrm{M}}) = \frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}} - \alpha}\overline{p}.$$

The seller finds it optimal to choose $p = \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$, rather than $\hat{p}(\omega^{\mathrm{M}})$, if

$$\begin{aligned}
&\mathbb{E}(v) - \frac{\chi - \gamma}{\beta} \geq \frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}} - \alpha}\overline{p}\overline{D}(\overline{p}) \\
\Leftrightarrow\quad & \frac{1 - \delta\chi}{\alpha}\omega^{\mathrm{M}} \geq \frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}} - \alpha}\overline{p}\overline{D}(\overline{p}) \\
\Leftrightarrow\quad & \omega^{\mathrm{M}} \geq \frac{\alpha}{1 - \delta\chi}\overline{p}\overline{D}(\overline{p}) + \frac{\alpha}{\delta\beta} \\
\Leftrightarrow\quad & \frac{\alpha}{1 - \delta\chi}\left(\mathbb{E}(v) - \frac{\chi - \gamma}{\beta}\right) \geq \frac{\alpha}{1 - \delta\chi}\overline{p}\overline{D}(\overline{p}) + \frac{\alpha}{\delta\beta} \\
\Leftrightarrow\quad & \beta \geq \frac{1 - \gamma\delta}{\delta}\frac{1}{\mathbb{E}(v) - \overline{p}\overline{D}(\overline{p})} = \ddot{\beta}
\end{aligned}$$

Thus, an equilibrium exists in this sub-case if and only if $\beta \geq \ddot{\beta}$ and $\omega^{\mathrm{M}} > g^{\beta}(\overline{p})$.

■

Lemma OA.9. *For any $\beta > \underline{\beta}$, there exists $\overline{\chi} > \gamma$ such that for any cap of market growth rate $\chi \in (\gamma, \overline{\chi})$, there is a unique finite Markov perfect equilibrium outcome $\mathbf{z}^{\mathrm{M}} = (r^{\mathrm{M}}, \sigma^{\mathrm{M}}, \omega^{\mathrm{M}}, p^{\mathrm{M}}, \{m_t^{\mathrm{M}}\})$ in which $p^{\mathrm{M}} = r^{\mathrm{M}} = \mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$, $\sigma^{\mathrm{M}} = \frac{\chi-\gamma}{\beta}$, $\omega^{\mathrm{M}} = \frac{\alpha}{1-\delta\chi}\left(\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}\right)$, and $m_t^{\mathrm{M}} = \chi^t$ for all $t \geq 0$.*

Proof. It suffices to show that given any $\beta > \underline{\beta}$, there exists $\overline{\chi} > \gamma$ such that for $\chi \in (\gamma, \overline{\chi})$, the conditions in Lemma OA.7 are violated and the conditions in Lemma OA.8 are satisfied.

First, we consider the three cases in Lemma OA.7. The first case of $\beta \leq \underline{\beta}$ is ruled out since we are considering $\beta > \underline{\beta}$. Now consider equilibria in the second case in Lemma OA.7, which occurs when $\beta \in (\underline{\beta}, \overline{\beta})$ and $\chi \geq \beta\left(\mathbb{E}(v) - \frac{1-\gamma\delta}{\alpha} g^{\beta}(p^{\beta})\right) + \gamma$. Notice that $\chi \geq \beta\left(\mathbb{E}(v) - \frac{1-\gamma\delta}{\alpha} g^{\beta}(p^{\beta})\right) + \gamma$ will be violated for χ close to γ if $\mathbb{E}(v) - \frac{1-\gamma\delta}{\alpha} g^{\beta}(p^{\beta}) > 0$, which is equivalent to

$$\frac{\delta\beta}{1-\gamma\delta} > \frac{1}{\mathbb{E}(v)}\left(1 + \frac{p^{\beta}\overline{D}(p^{\beta})}{\int_{p^{\beta}}^{\infty} \overline{D}(v)\,\mathrm{d}v}\right). \tag{OA.23}$$

By the definition of p^{β}, we have $\frac{\delta\beta}{1-\gamma\delta} = \frac{1}{\int_{p^{\beta}}^{\infty} \overline{D}(v)\,\mathrm{d}v}$. Thus, (OA.23) is equivalent to

$$\mathbb{E}(v) > p^{\beta}\overline{D}(p^{\beta}) + \int_{p^{\beta}}^{\infty} \overline{D}(v)\,\mathrm{d}v.$$

This inequality is always satisfied because $p^{\beta}\overline{D}(p^{\beta}) + \int_{p^{\beta}}^{\infty} \overline{D}(v)\,\mathrm{d}v < \int_{0}^{p^{\beta}} \overline{D}(v)\,\mathrm{d}v + \int_{p^{\beta}}^{\infty} \overline{D}(v)\,\mathrm{d}v = \mathbb{E}(v)$. For $\beta > \underline{\beta}$, we know that p^{β} is bounded away from 0. Thus, there exists $\overline{\chi}_1 > \gamma$ such that when $\chi \in (\gamma, \overline{\chi}_1)$, there is no equilibrium outcome in this case.

Now, consider equilibria in the third case in Lemma OA.7, which occurs when $\beta = \overline{\beta}$ and

$$\omega^{\mathrm{M}} \in \left[g^{\beta}(\overline{p}), \min\left\{\frac{\alpha}{\delta\beta}\left(1 - \frac{\overline{p}\overline{D}(\overline{p})}{\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}}\right)^{-1}, \frac{\alpha}{\delta\beta}\left(1 + \frac{\overline{p}\overline{D}(\overline{p})}{\int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v - \frac{\chi-\gamma}{\beta}}\right)\right\}\right].$$

To show that there is no equilibria in this case when χ is close to γ, it suffices to show that

$$\lim_{\chi\to\gamma} \frac{\alpha}{\delta\beta}\left(1 - \frac{\overline{p}\overline{D}(\overline{p})}{\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}}\right)^{-1} < g^{\beta}(\overline{p}),$$

i.e.,

$$\left(1 - \frac{\overline{p}\overline{D}(\overline{p})}{\mathbb{E}(v)}\right)^{-1} < \left(1 + \frac{\overline{p}\overline{D}(\overline{p})}{\int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v}\right).$$

This inequality is equivalent to

$$\mathbb{E}(v) > \overline{p}\overline{D}(\overline{p}) + \int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v$$

which is always satisfied because

$$\overline{p}\overline{D}(\overline{p}) + \int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v < \int_{0}^{\overline{p}} \overline{D}(v)\,\mathrm{d}v + \int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v = \mathbb{E}(v).$$

Hence, there exists $\overline{\chi}_2 > \gamma$ such that no equilibrium outcome exists in this case when $\chi \in (\gamma, \overline{\chi}_2)$.

We have ruled out possible equilibrium outcomes with a non-binding cap on the market growth rate when χ is close to γ. Next, we show that when χ is close to γ, the conditions in at least one of the two cases of Lemma OA.8 are satisfied. Let

$$\check{\beta} := \frac{1-\gamma\delta}{\delta}\frac{1}{\mathbb{E}(v)}\left(1 + \frac{\overline{p}\overline{D}(\overline{p})}{\int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v}\right).$$

Note that $\check{\beta} \in (\underline{\beta}, \overline{\beta})$. We consider two cases: $\beta \geq \check{\beta}$ and $\beta < \check{\beta}$.

Case 1: $\beta \geq \breve{\beta}$.

When $\chi \to \gamma$, we have $\omega^{\mathrm{M}} \to \breve{\omega} := \frac{\alpha}{1-\delta\gamma}\mathbb{E}(v)$. In this case, $\beta \geq \breve{\beta}$ implies $\breve{\omega} \geq g^{\beta}(\overline{p})$. Note that

$$\frac{\partial \omega^{\mathrm{M}}}{\partial \chi} = \frac{\alpha}{1-\chi\delta}\left[\frac{\delta}{1-\chi\delta}\left(\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}\right) - \frac{1}{\beta}\right].$$

At $\chi = \gamma$, the derivative is

$$\left.\frac{\partial \omega^{\mathrm{M}}}{\partial \chi}\right|_{\chi=\gamma} = \frac{\alpha}{1-\gamma\delta}\left(\frac{\delta}{1-\gamma\delta}\mathbb{E}(v) - \frac{1}{\beta}\right) > 0$$

where the inequality follows from the fact that $\beta > \underline{\beta}$. Hence, there exists $\overline{\chi}_3 > \gamma$ such that $\omega^{\mathrm{M}} > g^{\beta}(\overline{p})$ for $\chi \in (\gamma, \overline{\chi}_3)$. Note that $\ddot{\beta} < \breve{\beta}$ is equivalent to

$$\mathbb{E}(v) > \overline{p}\overline{D}(\overline{p}) + \int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v$$

which always holds because $\overline{p}\overline{D}(\overline{p}) + \int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v < \int_{0}^{\overline{p}} \overline{D}(v)\,\mathrm{d}v + \int_{\overline{p}}^{\infty} \overline{D}(v)\,\mathrm{d}v = \mathbb{E}(v)$. Thus, $\beta \geq \breve{\beta} > \ddot{\beta}$. According to Lemma OA.8, if $\beta \geq \breve{\beta}$, $\mathbf{z}^{\mathrm{M}}$ is an equilibrium for $\chi \in (\gamma, \overline{\chi}_3)$.

Case 2: $\beta < \breve{\beta}$.

Consider the equilibrium outcome $\mathbf{z}^{\mathrm{M}}$ in Lemma OA.8. In this case, $\beta < \breve{\beta}$ implies $\breve{\omega} < g^{\beta}(\overline{p})$. So for χ close to γ, $\omega^{\mathrm{M}} < g^{\beta}(\overline{p})$. According to Lemma OA.8, $\mathbf{z}^{\mathrm{M}}$ is an equilibrium with a binding cap on the market growth rate if $\beta \geq \underline{\beta}$ and

$$\mathbb{E}(v) - \frac{\chi-\gamma}{\beta} \geq \frac{\delta\beta\omega^{\mathrm{M}}}{\delta\beta\omega^{\mathrm{M}}-\alpha}(g^{\beta})^{-1}(\omega^{\mathrm{M}})\overline{D}((g^{\beta})^{-1}(\omega^{\mathrm{M}})). \tag{OA.24}$$

For (OA.24) to be satisfied for χ close to γ, we need

$$\mathbb{E}(v) > \frac{\delta\beta\breve{\omega}}{\delta\beta\breve{\omega}-\alpha}(g^{\beta})^{-1}(\breve{\omega})\overline{D}((g^{\beta})^{-1}(\breve{\omega})).$$

In this case, since $\breve{\omega} < g^{\beta}(\overline{p})$, we must have $\breve{\omega} = g^{\beta}\left(\left(1-\frac{\alpha}{\delta\beta\breve{\omega}}\right)\hat{p}(\breve{\omega})\right)$ which implies

$$\hat{p}(\breve{\omega}) = \frac{\delta\beta\breve{\omega}}{\delta\beta\breve{\omega}-\alpha}(g^{\beta})^{-1}(\breve{\omega}), \quad \text{and} \quad v^{-1}(\hat{p}(\breve{\omega})) = \left(1-\frac{\alpha}{\delta\beta\breve{\omega}}\right)\hat{p}(\breve{\omega}).$$

Thus,

$$\begin{aligned}
\frac{\delta\beta\breve{\omega}}{\delta\beta\breve{\omega}-\alpha}(g^{\beta})^{-1}(\breve{\omega})\overline{D}((g^{\beta})^{-1}(\breve{\omega})) &= \hat{p}(\breve{\omega})\overline{D}\left(\left(1-\frac{\alpha}{\delta\beta\breve{\omega}}\right)\hat{p}(\breve{\omega})\right) \\
&= \hat{p}(\breve{\omega})\overline{D}(v^{-1}(\hat{p}(\breve{\omega}))) \\
&= \mathbb{E}[v|v \geq v^{-1}(\hat{p}(\breve{\omega}))] \cdot \overline{D}\left(v^{-1}(\hat{p}(\breve{\omega}))\right) \\
&= \int_{v^{-1}(\hat{p}(\breve{\omega}))}^{\infty} v\overline{D}(\mathrm{d}v) \\
&< \mathbb{E}(v).
\end{aligned}$$

where the third equality holds by definition of $v^{-1}(\cdot)$ and the inequality holds because $\beta > \underline{\beta}$ implies $\breve{\omega} > \alpha/\delta\beta$, which in turn implies $v^{-1}(\hat{p}(\breve{\omega})) > 0$. Therefore, if $\beta \in (\underline{\beta}, \breve{\beta})$, there exists $\overline{\chi}_4$ such that for $\chi \in (\gamma, \overline{\chi}_4)$, $\mathbf{z}^{\mathrm{M}}$ is an equilibrium outcome.

In conclusion, let $\overline{\chi} = \min\{\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3, \overline{\chi}_4\}$. For $\chi \in (\gamma, \overline{\chi})$, the equilibrium outcome $\mathbf{z}^{\mathrm{M}}$ described in the lemma is the unique equilibrium outcome. ■

Proof of Proposition 7. This proposition immediately follows from Lemma OA.9. ■

Proof of Proposition 8. According to Lemma OA.9, for any $\underline{\beta} < \beta_1 < \beta_2 < \overline{\beta}$ and any $\chi \in (\gamma, \min\{\overline{\chi}^{\beta_1}, \overline{\chi}^{\beta_2}\})$, the unique Markov perfect equilibrium outcome under $\beta \in \{\beta_1, \beta_2\}$ is such that the sales revenue is $\mathbb{E}(v) - \frac{\chi-\gamma}{\beta}$ in each period and the total surplus in period t is $S_t^{\mathrm{M}}(\beta;\chi) = \chi^t \mathbb{E}(v)$. Since the sales revenue increases with the market feedback, the sellers are better off when the market feedback is higher. Hence, $\mathbf{x}^{\mathrm{M}}(\beta_1;\chi) \not\succ \mathbf{x}^{\mathrm{M}}(\beta_2;\chi)$. In addition, in the baseline model without a cap on market growth, total surplus in each period is $S_t^{\mathrm{M}}(\beta;\infty) = \frac{(1-\gamma\delta)}{\alpha} g^{\beta}(p^{\beta})$ which is strictly lower than $\mathbb{E}(v)$ and decreases in β for $\beta \in (\underline{\beta}, \overline{\beta})$. Therefore, $S_t^{\mathrm{M}}(\beta_1;\chi) = S_t^{\mathrm{M}}(\beta_2;\chi) = \chi^t \mathbb{E}(v) > S_t^{\mathrm{M}}(\beta_1;\infty) > S_t^{\mathrm{M}}(\beta_2;\infty)$.

■

Proof of Proposition 9. Suppose $\beta \in (\underline{\beta}, \overline{\beta})$. When there is a cap $\chi \in (\gamma, \min\{\overline{\chi}^{\beta_1}, \overline{\chi}^{\beta_2}\})$ on the market growth rate, the market size grows at a rate of χ in equilibrium, according to Lemma OA.9. In comparison, when there is no cap on the growth rate, the market size grows at a rate of γ in equilibrium, according to Theorem 1. Since $\chi > \gamma > 1$, the equilibrium market size when there is a cap will become exponentially larger than the market size when there is no cap. With a cap on growth, the seller's normalized surplus, the intermediary's normalized continuation payoff and the normalized consumer surplus in each period are all strictly positive. Hence, we must have $\mathbf{x}_t^{\mathrm{M}}(\beta;\chi) \succ \mathbf{z}_t^{\mathrm{M}}(\beta)$ for t large enough, due to the exponentially larger market size when there is a cap on growth. ■

Proof of Proposition 10. According to Lemma OA.9, for $\beta > \underline{\beta}$ and $\chi \in (\gamma, \overline{\chi}^{\beta})$, the total surplus is $S_t^{\mathrm{M}}(\beta;\chi) = \chi^t \mathbb{E}(v)$, which is strictly decreasing in χ. ■

OA.3.4 Proof of Proposition 11

First, notice that by similar arguments, an analogous version of Lemma 1 can be established. As a result, if $(\pi^{\mathrm{M}}, \omega^{\mathrm{M}})$ is a finite Markov perfect equilibrium payoff, then it must be that $\pi^{\mathrm{M}} = (1-\alpha)/\alpha\omega^{\mathrm{M}}$ and that

$$\omega^{\mathrm{M}} = \sup_{p \geq 0, D \in \mathcal{D}} \left[\alpha p D(p) + \delta \left(\gamma + \beta \int_p^{\infty} D(v)\,\mathrm{d}v\right) \omega^{\mathrm{M}}\right] = \max_{p \geq 0} \mathbf{W}(p|\omega^{\mathrm{M}}).$$

Suppose first that $\beta < \overline{\beta}$. Then it follows that $p = \mathbb{E}[v]$ and $\omega^{\mathrm{M}} = \alpha\mathbb{E}[v]/(1-\gamma\delta)$. To see this, notice that by Lemma OA.1, the maximizer of $\mathbf{W}(p|\omega^{\mathrm{M}})$ is either $p = 0$ or $p = \mathbb{E}[v]$. Moreover, suppose that $\omega^{\mathrm{M}} \geq \alpha/\delta\beta$, then

$$\mathbf{W}(0|\omega^{\mathrm{M}}) = \delta(\gamma + \beta\mathbb{E}[v])\omega^{\mathrm{M}}$$

and

$$\mathbf{W}(\mathbb{E}[v]|\mathrm{M}) = \alpha\mathbb{E}[v] + \gamma\delta\omega^{\mathrm{M}}.$$

Now suppose that $\mathbf{W}(0|\omega^{\mathrm{M}}) \geq \mathbf{W}(\mathbb{E}[v]|\omega^{\mathrm{M}})$ for all $p \geq 0$. This then implies that

$$\omega^{\mathrm{M}} = \delta(\gamma + \beta\mathbb{E}[v])\omega^{\mathrm{M}} < \delta(\gamma + \underline{\beta}\mathbb{E}[v])\omega^{\mathrm{M}} = \omega^{\mathrm{M}},$$

a contradiction. Thus, it must be that $\mathbf{W}(\mathbb{E}[v]|\omega^{\mathrm{M}}) = \omega^{\mathrm{M}} = \alpha\mathbb{E}[v]/(1-\gamma\delta)$. As a result, whenever $\beta < \underline{\beta}$, there is a unique finite Markov perfect equilibrium outcome in which $\omega^{\mathrm{M}} = \alpha\mathbb{E}[v]/(1-\gamma\delta)$.

Meanwhile, suppose that $\beta = \underline{\beta}$. Then $\mathbf{W}(0|\omega^{\mathrm{M}}) = 1$. Therefore, if $\omega^{\mathrm{M}} < \alpha/\delta\beta$, then

$$\omega^{\mathrm{M}} = \alpha\mathbb{E}[v] + \gamma\delta\omega^{\mathrm{M}}$$

and hence

$$\omega^{\mathrm{M}} = \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta} = \frac{\alpha}{\delta\beta},$$

a contradiction. Consequently, it must be that $\omega^{\mathrm{M}} \geq \alpha/\delta\beta$ and hence $p = 0$ maximizes $\mathbf{W}(p|\omega^{\mathrm{M}})$, but this implies that prices are zero in each period and hence $\omega^{\mathrm{M}} = 0$, a contradiction. Together, there is no finite Markov perfect equilibrium.

Lastly, suppose that $\beta > \underline{\beta}$. Then if $\omega^{\mathrm{M}} \geq \alpha/\delta\beta$,

$$\omega^{\mathrm{M}} = \mathbf{W}(0|\omega^{\mathrm{M}}) = \delta(\gamma + \beta\mathbb{E}[v])\omega^{\mathrm{M}} > \omega^{\mathrm{M}},$$

a contradiction, while if $\omega^{\mathrm{M}} < \alpha\delta\beta$, then

$$\omega^{\mathrm{M}} = \alpha\mathbb{E}[v] + \gamma\delta\omega^{\mathrm{M}}$$

and hence

$$\omega^{\mathrm{M}} = \frac{\alpha\mathbb{E}[v]}{1-\gamma\delta} > \frac{\alpha}{\delta\beta},$$

a contradiction. Together, there is no finite Markov perfect equilibrium either. This completes the proof. ■

OA.4 Omitted Proofs for Section 7

OA.4.1 Proof of Proposition 12

Similar to the baseline model, the one-shot deviation principle holds in this setting, and an analog of Lemma 2 can be established as in Lemma OA.10.

Lemma OA.10. *A finite Markov perfect equilibrium is characterized by a tuple $(\omega^{\mathrm{M}}, p^{\mathrm{M}}, \mathbf{D}^{\mathrm{M}})$ with $\omega^{\mathrm{M}}, p^{\mathrm{M}} \in [0,\infty)$ and $\mathbf{D}^{\mathrm{M}} : \mathbb{R}_+ \to \mathcal{D}$ that satisfy the following conditions:*

$$\omega^{\mathrm{M}} = \sup_{D\in\mathcal{D}} \left[\alpha p^{\mathrm{M}} D(p^{\mathrm{M}}) + \delta\omega^{\mathrm{M}} \cdot f\left(\int_{p^{\mathrm{M}}}^{\infty} D(v)\,\mathrm{d}v\right)\right], \tag{OA.25}$$

$$p^{\mathrm{M}}\mathbf{D}^{\mathrm{M}}(p^{\mathrm{M}}|p^{\mathrm{M}}) \geq p\mathbf{D}^{\mathrm{M}}(p|p), \tag{OA.26}$$

for all $p \geq 0$,

$$\alpha p\mathbf{D}^{\mathrm{M}}(p|p) + \delta\omega^{\mathrm{M}} \cdot f\left(\int_p^{\infty} \mathbf{D}^{\mathrm{M}}(v|p)\,\mathrm{d}v\right) \geq \alpha p D(p) + \delta\omega^{\mathrm{M}} \cdot f\left(\int_p^{\infty} D(v)\,\mathrm{d}v\right), \tag{OA.27}$$

for all $p \geq 0$ and for all $D \in \mathcal{D}$. Furthermore, for any Markov perfect equilibrium $(\omega^{\mathrm{M}}, p^{\mathrm{M}}, \mathbf{D}^{\mathrm{M}})$, its outcome is constant over time and is given by $(r^{\mathrm{M}}, p^{\mathrm{M}}, \sigma^{\mathrm{M}}, \omega^{\mathrm{M}})$, where $r^{\mathrm{M}} := p^{\mathrm{M}}\mathbf{D}^{\mathrm{M}}(p^{\mathrm{M}}|p^{\mathrm{M}})$, $\sigma^{\mathrm{M}} := \int_{p^{\mathrm{M}}}^{\infty} \mathbf{D}^{\mathrm{M}}(v|p^{\mathrm{M}})\,\mathrm{d}v$.

Let $\hat{\xi}(p|\omega)$ be the solution of

$$\alpha p = \delta\omega(p-\xi)f'\left(\int_{\xi}^{\infty}\overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi)\right),$$

and define $\tilde{\xi}(p|\omega) := \max\{0, \hat{\xi}(p|\omega)\}$.

Lemma OA.11. *For any $\alpha, \delta \in (0,1), \omega \geq 0$, any twice differentiable, increasing and concave function $f : \mathbb{R}_+ \to \mathbb{R}_+$, and for any $p \geq 0$,*

$$\Delta^f(p|\omega) := \underset{D\in\mathcal{D}}{\operatorname{argmax}}\left[\alpha p D(p) + \delta\omega \cdot f\left(\int_p^{\infty} D(v)\,\mathrm{d}v\right)\right]$$

is nonempty. Moreover, for any $D \in \Delta^f(p|\omega)$,

$$D(v) = \overline{D}(\xi(p|\omega)), \tag{OA.28}$$

for all $v \in [\xi(p|\omega), p]$ and

$$\int_{\xi(p|\omega)}^{\infty} D(v)\,\mathrm{d}v = \int_{\xi(p|\omega)}^{\infty}\overline{D}(v)\,\mathrm{d}v, \tag{OA.29}$$

where $\xi(p|\omega) := \max\left\{\tilde{\xi}(p|\omega), v^{-1}(p)\right\}$.

Proof. Arguments similar to the proof of Lemma 2 imply that the intermediary's problem can be simplified to

$$\max_{\xi\in[v^{-1}(p),p]} \alpha p\overline{D}(\xi) + \delta\omega f\left(\int_{\xi}^{\infty}\overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi)\right), \tag{OA.30}$$

which, by continuity of $\overline{D}$, has a solution. This implies that $\Delta^f(p|\omega)$ is nonempty. Moreover, the derivative of the objective function in (OA.30) is

$$\alpha p\overline{D}(\xi) - \delta\omega(p-\xi)f'\left(\int_{\xi}^{\infty}\overline{D}(v)\,\mathrm{d}v - (p-\xi)\overline{D}(\xi)\right)\overline{D}(\xi).$$

which is decreasing in ξ and negative when $\xi = p$, indicating that the intermediary's problem is concave in ξ and the solution $\xi(p|\omega)$ of (OA.30) is given by $\xi(p|\omega) := \max\left\{\tilde{\xi}(p|\omega), v^{-1}(p)\right\}$. This in turn implies that any $\widehat{D} \in \mathcal{D}$ satisfying the condition given by the lemma must be in $\Delta^f(p|\omega)$. ■

It is convenient to define

$$\tilde{\beta}(p|\omega) := f'\left(\int_{\xi(p|\omega)}^{\infty}\overline{D}(v)\,\mathrm{d}v - (p-\xi(p|\omega))\overline{D}(\xi(p|\omega))\right).$$

Lemma OA.12. *Given any $\alpha, \delta \in (0,1)$, $\omega \geq 0$ and $\beta \geq 0$, for any $\eta > 0$, let*

$$\mathcal{P}(\eta) := \left\{p = \underset{p\geq 0}{\operatorname{argmax}}\, p\mathbf{D}^f(p|p) : \ \mathbf{D}^f \text{ is a selection of } \Delta^f(\cdot|\omega) \text{ for some } f \in \mathcal{F}_2(\beta,\eta)\right\}.$$

Then, $\lim_{\eta\to 0}\sup_{p\in\mathcal{P}(\eta)}|p-\tilde{p}| = 0$, where $\tilde{p}$ is the unique value such that

$$v^{-1}(\tilde{p}) \leq \left(1 - \frac{\alpha}{\delta\beta\omega}\right)^{+}\tilde{p} \leq \overline{p}, \tag{OA.31}$$

with at least one inequality binding. Furthermore, if $\delta\beta\omega \leq \alpha$, we have $\mathcal{P}(\eta) = \{\tilde{p}\}$ for any η; if $\delta\beta\omega > \alpha$ and $v^{-1}(\tilde{p}) = (1 - \frac{\alpha}{\delta\beta\omega})\tilde{p} < \overline{p}$, there exists $\overline{\eta} > 0$ such that $\mathcal{P}(\overline{\eta}) = \{\tilde{p}\}$, $\tilde{\beta}(\tilde{p}|\omega) = f'(0)$ and $\int_{\tilde{p}}^{\infty}\mathbf{D}^f(v|\tilde{p})\,\mathrm{d}v = 0$ for any $f \in \mathcal{F}_2(\beta,\overline{\eta})$ and any selection $\mathbf{D}^f$ of $\Delta^f(\cdot|\omega)$.

Proof. Given any $f \in \mathcal{F}_2(\beta, \eta)$, by Lemma OA.11, for any selection $\mathbf{D}^f$ of $\Delta^f(\cdot|\omega)$,

$$p\mathbf{D}^f(p|p) = p\overline{D}(\xi(p|\omega)) = p\overline{D}\left(\max\left\{\left(1 - \frac{\alpha}{\delta\tilde{\beta}(p|\omega)\omega}\right)^+ p, v^{-1}(p)\right\}\right)$$
$$= \min\left\{p\overline{D}\left(\left(1 - \frac{\alpha}{\delta\tilde{\beta}(p|\omega)\omega}\right)^+ p\right), p\overline{D}(v^{-1}(p))\right\},$$

where the last equality follows from the fact that $\overline{D}$ is strictly decreasing.

Notice that by definition, we have $\tilde{\beta}(p|\omega) \in [f'(0), f'(\mathbb{E}(v)]$. Therefore, if $\|f''\| < \eta$ and as η goes to 0, we must have $\tilde{\beta}(p|\omega)$ converge to $f'(0) = \beta$ for all p. This implies that as η goes to 0, the seller's optimal price must converge to the price $\tilde{p}$ that solves

$$\min\left\{p\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right)^+ p\right), p\overline{D}(v^{-1}(p))\right\}, \tag{OA.32}$$

which has exactly the same form as the seller's problem in the baseline model with a market feedback level at β. Thus, the properties of $\tilde{p}$ stated in the lemma follows directly from Lemma 4.

Furthermore, when $\delta\beta\omega \leq \alpha$, since $f'' < 0$, we must have $\delta\tilde{\beta}(p|\omega)\omega \leq \alpha$ for any p. Hence, the seller's problem is to maximize

$$p\overline{D}(v^{-1}(p)),$$

which is the same as the problem in the baseline model. So the optimal price must be $\tilde{p}$ for any f with $f'(0) = \beta$. Thus, $\mathcal{P}(\eta) = \{\tilde{p}\}$ for any $\eta > 0$.

On the other hand, if $\delta\beta\omega > \alpha$ and $v^{-1}(\tilde{p}) = (1 - \frac{\alpha}{\delta\beta\omega})^+\tilde{p} < \overline{p}$, for η small enough, we have $\delta\tilde{\beta}(p|\omega)\omega > \alpha$ for all p, so

$$p\overline{D}\left(\left(1 - \frac{\alpha}{\delta\tilde{\beta}(p|\omega)\omega}\right)^+ p\right) = p\overline{D}\left(\left(1 - \frac{\alpha}{\delta\tilde{\beta}(p|\omega)\omega}\right) p\right). \tag{OA.33}$$

Its derivative with respect to p is

$$\overline{D}\left(\left(1 - \frac{\alpha}{\delta\tilde{\beta}(p|\omega)\omega}\right) p\right) + \left(1 - \frac{\alpha}{\delta\tilde{\beta}(p|\omega)\omega}\right) p\overline{D}'\left(\left(1 - \frac{\alpha}{\delta\tilde{\beta}(p|\omega)\omega}\right) p\right)$$
$$+ p^2\overline{D}'\left(\left(1 - \frac{\alpha}{\delta\tilde{\beta}(p|\omega)\omega}\right) p\right)\left(1 - \frac{\alpha}{\delta\tilde{\beta}(p|\omega)\omega}\right)',$$

which, as η goes to 0, converges to

$$\overline{D}\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right) p\right) + \left(1 - \frac{\alpha}{\delta\beta\omega}\right) p\overline{D}'\left(\left(1 - \frac{\alpha}{\delta\beta\omega}\right) p\right). \tag{OA.34}$$

Since $p\overline{D}(p)$ is strictly concave, (OA.34) must be strictly decreasing in p, positive for $p < \frac{\delta\beta\omega}{\delta\beta\omega - \alpha}\overline{p}$, and negative for $p > \frac{\delta\beta\omega}{\delta\beta\omega - \alpha}\overline{p}$. Hence, given any $\epsilon > 0$, for η small enough, (OA.33) is positive for $p < \frac{\delta\beta\omega}{\delta\beta\omega - \alpha}\overline{p} - \epsilon$ and negative for $p > \frac{\delta\beta\omega}{\delta\beta\omega - \alpha}\overline{p} + \epsilon$. Since $v^{-1}(\tilde{p}) = (1 - \frac{\alpha}{\delta\beta\omega})^+\tilde{p} < \overline{p}$, for η small enough, the seller's revenue function equals (OA.33) and must be strictly increasing for $p < \tilde{p}$, and it equals $p\overline{D}(v^{-1}(p))$ and is strictly decreasing for $p > \tilde{p}$. Therefore, the optimal price is $\tilde{p}$, i.e., $\mathcal{P}(\eta) = \{\tilde{p}\}$ for η small enough. According to Lemma OA.11, we have $\int_{\tilde{p}}^{\infty} \mathbf{D}^f(v|\tilde{p})\, dv = 0$ for any $\mathbf{D}^f$ that is a selection of $\Delta^f(\cdot|\omega)$, which also implies that $\tilde{\beta}(\tilde{p}|\omega) = f'(0)$. ■

To describe the equilibrium outcomes in this setting, we define

$$g^f(p) := \frac{\alpha}{\delta f'(0)} \left(1 + \frac{p\overline{D}(p)}{\int_p^\infty \overline{D}(v)\,\mathrm{d}v} \right),$$

for all $p \in [0, \overline{p}]$, and let

$$p^f := \inf\left\{ p \geq 0 \middle| \delta \left(f(0) + f'(0) \int_p^\infty \overline{D}(v)\,\mathrm{d}v \right) \geq 1 \right\}.$$

Lemma OA.13. *There exists a continuously decreasing function $h : (\underline{\beta}, \overline{\beta}) \to \mathbb{R}_+$ such that every $f \in \mathcal{F}_1 \cup [\bigcup_{\beta \in (\underline{\beta},\overline{\beta})} \mathcal{F}_2(\beta, h(\beta))]$ induces a unique finite Markov perfect equilibrium outcome. Furthermore, the following are equivalent:*

1. *$\mathbf{z}^{\mathrm{M}} = (r^{\mathrm{M}}, \sigma^{\mathrm{M}}, \omega^{\mathrm{M}}, p^{\mathrm{M}}, \{m_t^{\mathrm{M}}\})$ is a finite Markov perfect equilibrium outcome.*

2. $$p^{\mathrm{M}} = \begin{cases} \mathbb{E}[v], & \text{if } f \in \mathcal{F}_1 \\ v(p^f), & \text{if } f \in [\bigcup_{\beta \in (\underline{\beta},\overline{\beta})} \mathcal{F}_2(\beta, h(\beta))] \end{cases}.$$

   Moreover,

   $$\sigma^{\mathrm{M}} = \begin{cases} 0, & \text{if } f \in \mathcal{F}_1 \\ \int_{\overline{p}}^\infty \overline{D}(v)\,\mathrm{d}v - (p^{\mathrm{M}} - \overline{p})\overline{D}(\overline{p}), & \text{if } f \in [\bigcup_{\beta \in (\underline{\beta},\overline{\beta})} \mathcal{F}_2(\beta, h(\beta))] \end{cases};$$

   $$\omega^{\mathrm{M}} = \begin{cases} \frac{\alpha \mathbb{E}[v]}{1-\gamma\delta}, & \text{if } f \in \mathcal{F}_1 \\ g^f(p^f), & \text{if } f \in [\bigcup_{\beta \in (\underline{\beta},\overline{\beta})} \mathcal{F}_2(\beta, h(\beta))] \end{cases}; \quad r^{\mathrm{M}} = \begin{cases} \mathbb{E}[v], & \text{if } f \in \mathcal{F}_1 \\ \frac{(1-\gamma\delta)}{\alpha} g^f(p^f), & \text{if } f \in [\bigcup_{\beta \in (\underline{\beta},\overline{\beta})} \mathcal{F}_2(\beta, h(\beta))] \end{cases};$$

   and $m_t^{\mathrm{M}} = f(\sigma^{\mathrm{M}})^t$, for all $t \geq 1$.

Proof. We first show that given any $f'(0) < \overline{\beta}$, the equilibrium outcome $(r^{\mathrm{M}}, p^{\mathrm{M}}, \sigma^{\mathrm{M}}, \omega^{\mathrm{M}}, \{m_t^{\mathrm{M}}\})$ described in the statement of the theorem is indeed a finite Markov perfect equilibrium if either $f'(0) \in [0, \underline{\beta}]$ or $f \in \mathcal{F}_2(f'(0), \eta)$ for $\eta > 0$ small enough. To this end, we will show that for any such tuple, there exists $\mathbf{D}^{\mathrm{M}} : \mathbb{R}_+ \to \mathcal{D}$ such that $(\omega^{\mathrm{M}}, p^{\mathrm{M}}, \mathbf{D}^{\mathrm{M}})$ satisfies the conditions of Lemma OA.10.

Case 1: $f \in \mathcal{F}_1$.
In this case, we have $f'(0) \leq \underline{\beta}$. Consider any selection $\mathbf{D}^{\mathrm{M}}$ of $\Delta^f(\cdot|\omega^{\mathrm{M}})$. Since $p^{\mathrm{M}} = \mathbb{E}[v]$ and thus $v^{-1}(p^{\mathrm{M}}) = 0$, Lemma OA.12 implies that $p^{\mathrm{M}} \in \operatorname{argmax}_p p\mathbf{D}^{\mathrm{M}}(p|p)$, which establishes (OA.26). Meanwhile, Lemma OA.12 also implies that that

$$\int_{p^{\mathrm{M}}}^\infty \mathbf{D}^{\mathrm{M}}(v|p^{\mathrm{M}})\,\mathrm{d}v = 0.$$

Moreover, given $p^{\mathrm{M}} = \mathbb{E}[v]$, Lemma OA.11 implies that $\mathbf{D}^{\mathrm{M}}(p^{\mathrm{M}}|p^{\mathrm{M}}) = \overline{D}(0) = 1$. Together,

$$\begin{aligned}
\sup_{D \in \mathcal{D}} \left[\alpha p^{\mathrm{M}} D(p^{\mathrm{M}}) + \delta f\left(\int_{p^{\mathrm{M}}}^\infty D(v)\,\mathrm{d}v \right) \omega^{\mathrm{M}} \right] =& \alpha p^{\mathrm{M}} \mathbf{D}^{\mathrm{M}}(p^{\mathrm{M}}|p^{\mathrm{M}}) + \delta f\left(\int_{p^{\mathrm{M}}}^\infty \mathbf{D}^{\mathrm{M}}(v|p^{\mathrm{M}})\,\mathrm{d}v \right) \\
=& \alpha \mathbb{E}[v] + \delta f'(0) \omega^{\mathrm{M}} \\
=& \omega^{\mathrm{M}} \\
=& \frac{\alpha p^{\mathrm{M}} \mathbf{D}^{\mathrm{M}}(p^{\mathrm{M}}|p^{\mathrm{M}})}{1 - \delta f\left(\int_{p^{\mathrm{M}}}^\infty \mathbf{D}^{\mathrm{M}}(v|p)\,\mathrm{d}v \right)}
\end{aligned}$$

which establishes (OA.25) and (OA.27).

Case 2: $f \in \mathcal{F}_2(\beta, \eta)$ for some $\eta > 0$.
In this case, we have $f'(0) \in (\underline{\beta}, \overline{\beta})$. Take any selection $\mathbf{D}^{\mathrm{M}}$ of $\Delta^f(\cdot|\omega^{\mathrm{M}})$. Hence,

$$\left(1 - \frac{\alpha}{\delta f'(0)\omega^{\mathrm{M}}}\right)^+ p^{\mathrm{M}} = \left(1 - \frac{\alpha}{\delta f'(0)\omega^{\mathrm{M}}}\right) p^{\mathrm{M}} = v^{-1}(p^{\mathrm{M}}) = p^f < \overline{p},$$

which in turn implies that, by Lemma OA.12, for η small enough,

$$\int_{p^{\mathrm{M}}}^{\infty} \mathbf{D}^{\mathrm{M}}(v|p^{\mathrm{M}})\,\mathrm{d}v = 0$$

and therefore,

$$\omega^{\mathrm{M}} = \alpha p^{\mathrm{M}} \overline{D}(\xi(p^{\mathrm{M}}|\omega^{\mathrm{M}})) + f(0)\delta\omega^{\mathrm{M}} = \alpha p^{\mathrm{M}} \mathbf{D}^{\mathrm{M}}(p^{\mathrm{M}}|p^{\mathrm{M}}) + \delta f\left(\int_{p^{\mathrm{M}}}^{\infty} \mathbf{D}^{\mathrm{M}}(v|p^{\mathrm{M}})\,\mathrm{d}v\right)\omega^{\mathrm{M}},$$

which establishes (OA.25). Furthermore, since $p^f < \overline{p}$, $p^{\mathrm{M}} < \delta f'(0)\omega^{\mathrm{M}}\overline{p}/(\delta f'(0)\omega^{\mathrm{M}} - \alpha)$ and hence, for η small enough, p^{M} is the unique maximizer of $p\overline{D}(\xi(p|\omega^{\mathrm{M}}))$ according to Lemma OA.12. Thus, by Lemma OA.11, $(\omega^{\mathrm{M}}, p^{\mathrm{M}}, \mathbf{D}^{\mathrm{M}})$ satisfies (OA.26) and (OA.27). We define $h(\beta)$ to be the supermum of the values of η such that the above arguments are valid as being required in Lemma OA.12.

We now show that for any finite Markov perfect equilibrium, its outcome $(r^{\mathrm{M}}, p^{\mathrm{M}}, \sigma^{\mathrm{M}}, \omega^{\mathrm{M}}, \{m_t^{\mathrm{M}}\})$ must satisfy the conditions given by Lemma OA.13. By Lemma OA.10, if $(\omega^{\mathrm{M}}, p^{\mathrm{M}}, \mathbf{D}^{\mathrm{M}})$ satisfy (OA.25), (OA.26), and (OA.27) such that $r^{\mathrm{M}} = p^{\mathrm{M}}\mathbf{D}^{\mathrm{M}}(p^{\mathrm{M}}|p^{\mathrm{M}})$, $\sigma^{\mathrm{M}} = \int_{p^{\mathrm{M}}}^{\infty} \mathbf{D}^{\mathrm{M}}(v|p^{\mathrm{M}})\,\mathrm{d}v$ and $m_t^{\mathrm{M}} = f(\sigma^{\mathrm{M}})^t$. It follows immediately that $r^{\mathrm{M}}, \sigma^{\mathrm{M}}, \{m_t^{\mathrm{M}}\}$ satisfy the condition given by Lemma OA.13 if ω^{M} and p^{M} satisfy these conditions. Thus, it suffices to show that $\omega^{\mathrm{M}}, p^{\mathrm{M}}$ satisfy these conditions. Now consider three cases separately.

Case 1: $\omega^{\mathrm{M}} \leq \alpha/\delta f'(0)$.
In this case, by Lemma OA.12, it immediately follows that

$$\left(1 - \frac{\alpha}{\delta f'(0)\omega^{\mathrm{M}}}\right)^+ p^{\mathrm{M}} = 0 = v^{-1}(p^{\mathrm{M}})$$

and hence $p^{\mathrm{M}} = \mathbb{E}[v]$, which in turn, by (OA.25), implies that $\omega^{\mathrm{M}} = \alpha\mathbb{E}[v]/(1 - f'(0)\delta)$. For this to be consistent with $\omega^{\mathrm{M}} \leq \alpha/\delta f'(0)$, it must be that $f'(0) \leq \underline{\beta}$, i.e., $f \in \mathcal{F}_1$.

Case 2: $\omega^{\mathrm{M}} > \alpha/\delta f'(0)$ and

$$\left(1 - \frac{\alpha}{\delta f'(0)\omega^{\mathrm{M}}}\right) p^{\mathrm{M}} = v^{-1}(p^{\mathrm{M}}) < \overline{p}. \tag{OA.35}$$

In this case, Lemma OA.12 implies that, there exists $\overline{\eta}$ such that, if $\|f''\| < \overline{\eta}$, p^{M} is the unique optimal price for the seller, i.e., it satisfies (OA.26). Then, Lemma OA.11 implies that

$$\omega^{\mathrm{M}} = \delta\left(f(0) + f'(0)\int_{\left(1 - \frac{\alpha}{\delta f'(0)\omega^{\mathrm{M}}}\right)p^{\mathrm{M}}}^{\infty} \overline{D}(v)\,\mathrm{d}v\right)\omega^{\mathrm{M}},$$

and hence, together with (OA.31), it must be that $f'(0) \in [\underline{\beta}, \overline{\beta}]$ and

$$\left(1 - \frac{\alpha}{\delta f'(0)\omega^{\mathrm{M}}}\right) p^{\mathrm{M}} = p^f.$$

Meanwhile, since (OA.35) is equivalent to

$$\omega^{\mathrm{M}} = g^f\left(\left(1 - \frac{\alpha}{\delta f'(0)\omega^{\mathrm{M}}}\right) p^{\mathrm{M}}\right),$$

it must be that $\omega^{\mathrm{M}} = g^f(p^f)$ and hence $p^{\mathrm{M}} = v(p^f)$.

Case 3: $\omega^{\mathrm{M}} > \alpha/\delta f'(0)$ and (OA.35) is violated. In this case, Lemma OA.12 implies that, as $\|f''\|$ goes to zero

$$p^{\mathrm{M}} \to \frac{\delta f'(0)\omega^{\mathrm{M}}}{\delta f'(0)\omega^{\mathrm{M}} - \alpha}\overline{p} \quad \text{and} \quad \xi(p|\omega^{\mathrm{M}}) \to \overline{p}. \tag{OA.36}$$

In this case, Lemma OA.10 and Lemma OA.11 implies that

$$\delta\left[f'\left(\int_{\overline{p}}^{\infty}\overline{D}(v)\,\mathrm{d}v - (p^{\mathrm{M}} - \overline{p})\overline{D}(\overline{p})\right)(p^{\mathrm{M}} - \overline{p})\overline{D}(\overline{p}) + f\left(\int_{\overline{p}}^{\infty}\overline{D}(v)\,\mathrm{d}v - (p^{\mathrm{M}} - \overline{p})\overline{D}(\overline{p})\right)\right] \to 1,$$

or equivalently,

$$\delta\left[f'\left(\sigma(p^{\mathrm{M}})\right)(p^{\mathrm{M}} - \overline{p})\overline{D}(\overline{p}) + f\left(\sigma(p^{\mathrm{M}})\right)\right] \to 1, \tag{OA.37}$$

where $\sigma(p^{\mathrm{M}}) = \int_{\overline{p}}^{\infty}\overline{D}(v)\,\mathrm{d}v - (p^{\mathrm{M}} - \overline{p})\overline{D}(\overline{p})$.

Notice that as $\|f''\|$ goes to zero, $f'\left(\sigma^{\mathrm{M}}\right) \to f'(0)$ and $f\left(\sigma^{\mathrm{M}}\right) \to f(0) + f'(0)\cdot\sigma^{\mathrm{M}}$. Hence, if $f'(0) < \overline{\beta}$,

$$\begin{aligned}
\delta\left[f'\left(\sigma^{\mathrm{M}}\right)(p^{\mathrm{M}} - \overline{p})\overline{D}(\overline{p}) + f\left(\sigma^{\mathrm{M}}\right)\right] \to& \delta\left[f'(0)(p^{\mathrm{M}} - \overline{p})\overline{D}(\overline{p}) + f(0) + f'(0)\cdot\sigma^{\mathrm{M}}\right] \\
<& \delta\left(f(0) + \overline{\beta}\int_{\overline{p}}^{\infty}\overline{D}(v)\right) \\
=& 1
\end{aligned} \tag{OA.38}$$

which contradicts with equation (OA.37). Therefore, for $\|f''\|$ small enough, this case can only occur when $f'(0) \geq \overline{\beta}$.

Therefore, there exists a function $h : (\underline{\beta}, \overline{\beta}) \to \mathbb{R}_+$ such that every $f \in \mathcal{F}_1 \cup [\bigcup_{\beta\in(\underline{\beta},\overline{\beta})}\mathcal{F}_2(\beta, h(\beta))]$ induces a unique finite Markov perfect equilibrium outcome as described in this lemma. Finally, notice that the upper bound of $\|f''\|$ for the preceding arguments to be valid is bounded away from zero except for β arbitrarily close to $\overline{\beta}$. Hence, the function $h : (\underline{\beta}, \overline{\beta}) \to \mathbb{R}_+$ can be chosen to be continuously decreasing. ∎

Proof of Proposition 12. This proposition follows from Lemma OA.13. ∎

References

Milgrom, P. and I. Segal (2002): "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70, 583–601.