Online Appendix of
Distributions of Posterior Quantiles and Economic Applications

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OA.1 Properties of Distributions of Posterior Quantiles

OA.1.1 Proof of Corollary 2

For any $H \in \mathcal{H}_\tau$, Theorem 1 implies that

$$\overline{F}_0^\tau(\underline{\omega}) \leq \overline{F}_0^\tau(\overline{\omega}) \leq H(\underline{\omega}) \leq H(\overline{\omega}) \leq \underline{F}_0^\tau(\underline{\omega}) \leq \underline{F}_0^\tau(\overline{\omega}).$$

Therefore,

$$(\overline{F}_0^\tau(\underline{\omega})) - \underline{F}_0^\tau(\overline{\omega}))^+ \leq H(\overline{\omega}) - H(\underline{\omega}) \leq \underline{F}_0^\tau(\overline{\omega}) - \overline{F}_0^\tau(\underline{\omega}).$$

Conversely, for any $\eta \in [(\overline{F}_0^\tau(\underline{\omega})) - \underline{F}_0^\tau(\overline{\omega}))^+, (\underline{F}_0^\tau(\overline{\omega}) - \overline{F}_0^\tau(\underline{\omega}))]$, if $\underline{\omega} \leq F_0^{-1}(\tau)$, then let H be defined as

$$H(\omega) := \begin{cases} \underline{F}_0^\tau(\omega), & \text{if } \omega < (\underline{F}_0^\tau)^{-1}(\eta + \underline{F}_0^\tau(\underline{\omega})) \\ \eta + \underline{F}_0^\tau(\underline{\omega}), & \text{if } \omega \in [(\underline{F}_0^\tau)^{-1}(\eta + \underline{F}_0^\tau(\underline{\omega})), (\overline{F}_0^\tau)^{-1}((\eta + \underline{F}_0^\tau(\underline{\omega}))^+) \\ \overline{F}_0^\tau(\omega), & \text{if } \omega \geq (\overline{F}_0^\tau)^{-1}((\eta + \underline{F}_0^\tau(\underline{\omega}))^+). \end{cases},$$

for all $\omega \in \mathbb{R}$; if $\underline{\omega} > F_0^{-1}(\tau)$, then define H as

$$H(\omega) := \begin{cases} \underline{F}_0^\tau(\omega), & \text{if } \omega < (\underline{F}_0^\tau)^{-1}(\overline{F}_0^\tau(\overline{\omega}) - \eta) \\ \overline{F}_0^\tau(\overline{\omega}) - \eta, & \text{if } \omega \in [(\underline{F}_0^\tau)^{-1}(\overline{F}_0^\tau(\overline{\omega}) - \eta), (\overline{F}_0^\tau)^{-1}((\overline{F}_0^\tau(\overline{\omega}) - \eta)^+) \\ \overline{F}_0^\tau(\omega), & \text{if } \omega \geq (\overline{F}_0^\tau)^{-1}((\overline{F}_0^\tau(\overline{\omega}) - \eta)^+). \end{cases}.$$

In both cases, $H(\overline{\omega}) - H(\underline{\omega}) = \eta$ and $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$. By Theorem 1, $H \in \mathcal{H}_\tau$ as well. This completes the proof. ■

OA.1.2 Proof of Theorem 2

For any $F_0 \in \mathcal{F}$ and for any $\tau \in (0,1)$, let $\mathcal{M}_\tau^0(F_0)$ denote the collection of signals $\mu \in \mathcal{M}(F_0)$ such that

$$\mu(\{F \in \mathcal{F} | F^{-1}(\tau) = F^{-1}(\tau^+)\}) = 1.$$

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When there is no confusion, we write $\mathcal{M}_\tau^0(F_0)$ as $\mathcal{M}_\tau^0$. Note that for any $\mu \in \mathcal{M}_\tau^0$, $H^\tau(\omega|\mu, r) = H^\tau(\omega|\mu, r')$ for all $\omega \in \mathbb{R}$ and for all $r, r' \in \mathcal{R}$. Henceforth, we denote the distribution of posterior quantiles induced by $\mu \in \mathcal{M}_\tau^0$ as $H^\tau(\omega|\mu)$.

To prove Theorem 2, we first establish two lemmas that generalize Lemma 1 and Lemma 2 in the main text.

Lemma OA.1. *For any $\tau \in (0,1)$, and for any nondecreasing and right-continuous function $g : [0,1] \to [0,1]$, the following are equivalent:*

1. *For any $F_0 \in \mathcal{F}$ and for any $H \in \mathcal{F}$ such that $H \succeq g \circ F_0$ ($H \preceq g \circ F_0$, resp.), there exists $\mu \in \mathcal{M}_\tau^0(F_0)$ such that $H^\tau(\omega|\mu) = H(\omega)$.*

2. *For any $H \in \mathcal{F}$ such that $H \succeq g$ ($H \preceq g$, resp.), there exists $\tilde{\mu} \in \mathcal{M}_\tau^0(U)$ such that $H^\tau(\omega|\tilde{\mu}) = H(\omega)$.*

Proof. Since the CDF of the uniform distribution on $[0,1]$ is in $\mathcal{F}$ and since U is the identity function, 1 implying 2 follows immediately from the definition of $\tilde{r}_0$.

Conversely, suppose that 2 holds. Consider any $F_0 \in \mathcal{F}$ and any $H \in \mathcal{F}$ such that $H \succeq g \circ F_0$. We now show that there exists $\mu \in \mathcal{M}_\tau^0$ such that $H^\tau(\omega|\mu) = H(\omega)$. To this end, let $\widetilde{H}(q) := H(F_0^{-1}(q))$ for all $q \in \mathbb{R}$. Since g is nondecreasing, it must be that $\widetilde{H} \succeq g$. Therefore, there exists $\tilde{\mu} \in \mathcal{M}_\tau^0(U)$ such that

$$\widetilde{H}(q) = H^\tau(q|\tilde{\mu}) = \tilde{\mu}(\{F \in \mathcal{F} | F^{-1}(\tau) \leq q\}).$$

for all $q \in [0,1]$. Next, for any $F_0 \in \mathcal{F}$, define μ as

$$\mu(A) := \tilde{\mu}(\{F \in \mathcal{F} | F \circ F_0 \in A\}),$$

for all measurable $A \subseteq \mathcal{F}$. We claim that $\mu \in \mathcal{M}_\tau^0(F_0)$. Indeed, for any measurable $A \subseteq \mathcal{F}$, $\mu(A) = \tilde{\mu}(\{F \in \mathcal{F} | F \circ F_0 \in A\}) \geq 0$. Meanwhile, $\mu(\mathcal{F}) = \tilde{\mu}(\{F \in \mathcal{F} | F \circ F_0 \in \mathcal{F}\}) = \tilde{\mu}(\mathcal{F}) = 1$. Furthermore, for any measurable $A \subseteq \mathcal{F}$, let

$$F_0^{-1} \circ A := \{F_0^{-1} \circ F | F \in A\},$$

and note that $F \circ F_0 \in A$ if and only if $F \in F_0^{-1} \circ A$ for all $F \in \mathcal{F}$. Thus, for any disjoint collection of measurable sets $\{A_n\} \subseteq \mathcal{F}$,

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \tilde{\mu}\left(\left\{F \in \mathcal{F} \middle| F \circ F_0 \in \bigcup_{n=1}^{\infty} A_n\right\}\right) \\
&= \tilde{\mu}\left(\left\{F \in \mathcal{F} \middle| F \in F_0^{-1} \circ \bigcup_{n=1}^{\infty} A_n\right\}\right) \\
&= \sum_{n=1}^{\infty} \tilde{\mu}(F_0^{-1} \circ A_n) \\
&= \sum_{n=1}^{\infty} \tilde{\mu}(\{F \in \mathcal{F} | F \circ F_0 \in A_n\}) \\
&= \sum_{n=1}^{\infty} \mu(A_n).
\end{aligned}$$

Consequently, μ is indeed a probability measure on $\mathcal{F}$.

In addition, note that, for any $\omega \in \mathbb{R}$,

$$\int_{\mathcal{F}} F(\omega)\mu(\mathrm{d}F) = \int_{\mathcal{F}} F(F_0(\omega))\tilde{\mu}(\mathrm{d}F) = F_0(\omega),$$

which in turn implies that $\mu \in \mathcal{M}(F_0)$. Lastly, since F_0 has a support on an interval of $\mathbb{R}$, F_0 is strictly increasing on its support. As a result,

$$\mu(\{F \in \mathcal{F} | F^{-1}(\tau) = F^{-1}(\tau)\}) = \tilde{\mu}(\{F \in \mathcal{F} | F_0^{-1} \circ F^{-1}(\tau) = F_0^{-1} \circ F^{-1}(\tau^+)\}) = 1,$$

and hence $\mu \in \mathcal{M}_\tau^0(F_0)$.

As a result, for any $\omega \in \mathbb{R}$,

$$\begin{aligned} H(\omega) = \widetilde{H}(F_0(\omega)) =& \tilde{\mu}(\{F \in \mathcal{F} | F^{-1}(\tau) \leq F_0(\omega)\}) \\ =& \tilde{\mu}(\{F \in \mathcal{F} | F_0^{-1} \circ F^{-1}(\tau) \leq \omega\}) \\ =& \mu(\{F \in \mathcal{F} | F^{-1}(\tau) \leq \omega\}) \\ =& H^\tau(\omega|\mu), \end{aligned}$$

as desired. The case for $H \preceq g \circ F_0$ is analogous. This completes the proof. ■

To set the stage for the next lemma, we note that for any $\tau \in (0,1)$ and for any $\underline{\tau}, \overline{\tau}$ such that $0 < \underline{\tau} < \tau < \overline{\tau} < 1$, $\underline{U}^{\overline{\tau}} \preceq \overline{U}^{\underline{\tau}}$ whenever $\tau - \underline{\tau}$ and $\overline{\tau} - \tau$ are small enough. For any such $\underline{\tau}, \overline{\tau}$, let $\mathcal{I}^*_{\underline{\tau},\overline{\tau}} := \mathcal{I}(\underline{U}^{\overline{\tau}}, \overline{U}^{\underline{\tau}})$.

Lemma OA.2. *H is an extreme point of $\mathcal{I}^*_{\underline{\tau},\overline{\tau}}$ if and only if there exists $0 \leq \underline{x} \leq \overline{x} \leq \underline{\tau} \leq \tau \leq \overline{\tau} \leq \underline{y} \leq \overline{y}$; countable sets I, J; and sequences $\{\underline{x}_i, \overline{x}_i\}_{i \in I}, \{\underline{y}_j, \overline{y}_j\}_{j \in J} \subseteq \mathbb{R}$ such that $\underline{U}^{\overline{\tau}}(\overline{x}) = \overline{U}^{\underline{\tau}}(\underline{y})$; that $\underline{x} \leq \underline{x}_i \leq \overline{x}_i \leq \underline{x}_{i+1} \leq \overline{x} < \underline{y} \leq \underline{y}_j \leq \overline{y}_j \leq \underline{y}_{j+1} \leq \overline{y}$ for all $i \in I, j \in J$; and that*

$$H(\omega) = \begin{cases} 0, & \text{if } \omega < \underline{x} \\ \underline{U}^{\overline{\tau}}(\underline{x}_i), & \text{if } \omega \in [\underline{x}_i, \overline{x}_i) \\ \underline{U}^{\overline{\tau}}(\omega), & \text{if } \omega \in [\underline{x}, \overline{x}) \backslash \cup_{i \in I} [\underline{x}_i, \overline{x}_i) \\ \underline{U}^{\overline{\tau}}(\overline{x}), & \text{if } \omega \in [\overline{x}, \underline{y}) \\ \overline{U}^{\underline{\tau}}(\underline{y}_j), & \text{if } \omega \in [\underline{y}_j, \overline{y}_j) \\ \overline{U}^{\underline{\tau}}(\omega), & \text{if } \omega \in [\underline{y}, \overline{y}) \backslash \cup_{j \in J} [\underline{y}_j, \overline{y}_j) \\ 1, & \text{if } \omega \geq \overline{y} \end{cases}, \tag{OA.1}$$

for all $\omega \in \mathbb{R}$.

Proof. Embed $\mathcal{I}^*_{\underline{\tau},\overline{\tau}} \subseteq \mathcal{F}$ into the collection $L^1([0,1])$ of integrable functions on $[0,1]$. Note that $\mathcal{I}^*_\tau$ is a convex subset of a normed linear space $L^1([0,1])$. Consider any $H \in \mathcal{I}^*_\tau$ that takes the form of (OA.1), and any $\widehat{H} \in L^1([0,1])$ such that $\widehat{H}(\tilde{\omega}) \neq 0$ for some $\tilde{\omega} \in [0,1]$. Suppose that $H(\tilde{\omega}) \in \{\underline{U}^{\overline{\tau}}(\tilde{\omega}), \overline{U}^{\underline{\tau}}(\tilde{\omega})\}$. Then clearly either $H(\tilde{\omega}) + \widehat{H}(\tilde{\omega}) > \underline{U}^{\overline{\tau}}(\tilde{\omega})$ or $H(\tilde{\omega}) - \widehat{H}(\tilde{\omega}) < \overline{U}^{\underline{\tau}}(\tilde{\omega})$ and hence, either $H + \widehat{H} \notin \mathcal{I}^*_\tau$ or $H - \widehat{H} \notin \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$. Meanwhile, suppose that $\tilde{\omega} \in [\underline{x}_i, \overline{x}_i)$ for some $i \in I$ or $\tilde{\omega} \in [\underline{y}_j, \overline{y}_j)$ for some $j \in J$. If either $H + \widehat{H} \notin \mathcal{F}$ or $H - \widehat{H} \notin \mathcal{F}$, then clearly either $H + \widehat{H} \notin \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$ or $H - \widehat{H} \notin \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$. If, on the other hand, both $H + \widehat{H}$ and $H - \widehat{H}$ are in $\mathcal{F}$, then it must be that either $H(\omega) + \widehat{H}(\omega) = \underline{U}^{\overline{\tau}}(\underline{x}_i) + \widehat{H}(\tilde{\omega}) > \underline{U}^{\overline{\tau}}(\underline{x}_i)$ for all

$\omega \in [\underline{x}_i, \overline{x}_i)$, or $H(\omega) - \widehat{H}(\omega) = \overline{U}^{\underline{\tau}}(\overline{y}_j) - \widehat{H}(\tilde{\omega}) < \overline{U}^{\underline{\tau}}(\overline{y}_j)$, for all $\omega \in [\underline{y}_j, \overline{y}_j)$. Therefore, there must exist $\hat{\omega} \in \mathbb{R}$ such that either $H(\hat{\omega}) + \widehat{H}(\hat{\omega}) \notin \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$ or $H(\hat{\omega}) - \widehat{H}(\hat{\omega}) \notin \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$.

Conversely, suppose that $H \in \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$ does not take form of (OA.1). Then there exists $\underline{\omega} < \overline{\omega}$ and $\underline{\eta} < \overline{\eta}$ such that $H(\underline{\omega}^-) \leq \underline{\eta} \leq H(\underline{\omega})$, $H(\overline{\omega}^-) \leq \overline{\eta} \leq H(\overline{\omega})$; that $\overline{U}^{\underline{\tau}}(\overline{\omega}) \leq \underline{\eta} < \overline{\eta} \leq \underline{U}^{\overline{\tau}}(\underline{\omega})$; and that $\underline{\eta} < H(\omega) < \overline{\eta}$ for some $\omega \in (\underline{\omega}, \overline{\omega})$. Then, since the set of extreme points of nondecreasing functions that map from $[\underline{\omega}, \overline{\omega}]$ to $[\underline{\eta}, \overline{\eta}]$ must only take values in $\{\underline{\eta}, \overline{\eta}\}$ (see, for instance, lemma 2.7 of Börgers, 2015), there exists a non-zero, integrable function $\widetilde{H} : [\underline{\omega}, \overline{\omega}] \to [\underline{\eta}, \overline{\eta}]$ such that both $H + \widetilde{H}$ and $H - \widetilde{H}$ are nondecreasing, right-continuous functions from $[\underline{\omega}, \overline{\omega}]$ to $[\underline{\eta}, \overline{\eta}]$. As a result, for any $\omega \in [\underline{\omega}, \overline{\omega}]$, it must be that

$$\max\{H(\omega) + \widetilde{H}(\omega), H(\omega) - \widetilde{H}(\omega)\} \leq \overline{\eta} \leq \underline{U}^{\overline{\tau}}(\underline{\omega}) \leq \underline{U}^{\overline{\tau}}(\omega) \tag{OA.2}$$

and that

$$\min\{H(\omega) + \widetilde{H}(\omega), H(\omega) - \widetilde{H}(\omega)\} \geq \underline{\eta} \geq \overline{U}^{\underline{\tau}}(\overline{\omega}) \geq \overline{U}^{\underline{\tau}}(\omega), \tag{OA.3}$$

for all $\omega \in [\underline{\omega}, \overline{\omega}]$. Now let $\widehat{H} : [0,1] \to \mathbb{R}$ be defined as

$$\widehat{H}(\omega) := \begin{cases} \widetilde{H}(\omega), & \text{if } \omega \in [\underline{\omega}, \overline{\omega}] \\ 0, & \text{otherwise} \end{cases},$$

for all $\omega \in [0,1]$. Clearly, $\widehat{H} \in L^1([0,1])$. Moreover, for any $\omega \in [0,1]$, from (OA.2) and (OA.3), together with $H \in \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$, it follows that

$$\overline{U}^{\underline{\tau}}(\omega) \leq \min\{H(\omega) + \widehat{H}(\omega), H(\omega) - \widehat{H}(\omega)\} \leq \max\{H(\omega) + \widehat{H}(\omega), H(\omega) - \widehat{H}(\omega)\} \leq \underline{U}^{\overline{\tau}}(\omega),$$

for all $\omega \in [0,1]$. Meanwhile, since $\underline{\eta} \in [H(\underline{\omega}^-), H(\underline{\omega})]$ and $\overline{\eta} \in [H(\overline{\omega}^-), H(\overline{\omega})]$, it must be that

$$H(\omega) + \widehat{H}(\omega) = H(\omega) - \widehat{H}(\omega) = H(\omega) \leq H(\underline{\omega}^-) \leq \underline{\eta},$$

for all $\omega \leq \underline{\omega}$; while

$$H(\omega) + \widehat{H}(\omega) = H(\omega) - \widehat{H}(\omega) = H(\omega) \geq H(\overline{\omega}) \geq \overline{\eta},$$

for all $\omega \geq \overline{\omega}$. As a result, both $H + \widehat{H}$ and $H - \widehat{H}$ are nondecreasing and right-continuous. It then follows that $H + \widehat{H} \in \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$ and $H - \widehat{H} \in \mathcal{I}^*_{\underline{\tau},\overline{\tau}}$, and hence H is not an extreme point of $\mathcal{I}^*_{\underline{\tau},\overline{\tau}}$. This completes the proof. ■

Proof of Theorem 2. By Theorem 1,

$$H^0_\tau \subseteq \mathcal{H}_\tau = \mathcal{I}(\underline{F}^\tau_0, \overline{F}^\tau_0).$$

We now show that $\mathcal{I}(\underline{F}^{\tau+\varepsilon}_0, \overline{F}^{\tau-\varepsilon}_0) \subseteq \mathcal{H}^0_\tau$ for all $\varepsilon > 0$ small enough. To this end, first notice that since F_0 has full support on an interval in $\mathbb{R}$, $\mathcal{I}(\underline{F}^{\tau+\varepsilon}_0, \overline{F}^{\tau-\varepsilon}_0)$ is well-defined when ε is small enough. Then, notice that since the functions $q \mapsto \min\{q/\tau+\varepsilon, 1\}$ and $q \mapsto \max\{q-(\tau-\varepsilon)/1-(\tau-\varepsilon), 0\}$ are nondecreasing for $\varepsilon > 0$ small enough, Lemma OA.1 implies that it suffices to prove $\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon} \subseteq \mathcal{H}^0_\tau$. To show this, consider any $\varepsilon > 0$ small enough and consider any extreme point H of $\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon}$. By Lemma OA.2, H must take the form of (OA.1) for some $\underline{x}, \overline{x}, \underline{y}, \overline{y} \in [0,1]$ and countable sequences $\{\underline{x}_i, \overline{x}_i\}_{i \in I}$ and $\{\underline{y}_j, \overline{y}_j\}_{j \in J}$, such that

$\underline{x} \leq \underline{x}_i \leq \overline{x}_i \leq \underline{x}_{i+1} \leq \overline{x} < \underline{y} \leq \underline{y}_j \leq \overline{y}_j \leq \underline{y}_{j+1} \leq \overline{y}$ for all $i \in I, j \in J$. Now define two classes of distributions, $\{\underline{U}^\omega\}_{\omega\in[0,\overline{x}]}$ and $\{\overline{U}^\omega\}_{\omega\in[\underline{y},1]}$, as follows:

$$\underline{U}^\omega(x) := \begin{cases} 0, & \text{if } x < \omega \\ \frac{\overline{x}}{\underline{y}-(\tau+\varepsilon)+\overline{x}}, & \text{if } x \in [\omega, \tau+\varepsilon) \\ \frac{x-(\tau+\varepsilon)+\overline{x}}{1-\underline{y}+\overline{x}}, & \text{if } x \in [\tau+\varepsilon, \underline{y}) \\ 1, & \text{if } x \geq \underline{y} \end{cases}; \text{ and } \overline{U}^\omega(x) := \begin{cases} 0, & \text{if } x < \overline{x} \\ \frac{x-\overline{x}}{1-\underline{y}+\tau-\varepsilon-\overline{x}}, & \text{if } x \in [\overline{x}, \tau-\varepsilon) \\ \frac{\tau-\varepsilon-\overline{x}}{1-\underline{y}+\tau-\varepsilon-\overline{x}}, & \text{if } x \in [\overline{x}, \omega) \\ 1, & \text{if } x \geq \omega \end{cases}.$$

Note that since $\underline{U}^{\tau+\varepsilon}(\overline{x}) = \overline{U}^{\tau-\varepsilon}(\underline{y})$, it follows that $(1-\tau+\varepsilon)\overline{x} = (\tau+\varepsilon)(\underline{y}-\tau+\varepsilon)$, and hence, $\underline{U}^\omega(x) = \tau+\varepsilon$ for all $x \in [\omega, \underline{y}]$ and $\overline{U}^\omega(x) = \tau - \varepsilon$ for all $x \in [\overline{x}, \omega)$. As a result, $\mathbb{Q}^\tau(\underline{U}^\omega) = \{\omega\}$ for all $\omega \in [0, \overline{x}]$ and $\mathbb{Q}^\tau(\overline{U}^\omega) = \{\omega\}$ for all $\omega \in [\underline{y}, 1]$. Moreover, for any $i \in I$ and for any $j \in J$, let $\underline{U}^i$ and $\overline{U}^j$ be defined as

$$\underline{U}^i(x) := \frac{1}{\overline{x}_i - \underline{x}_i}\int_{\underline{x}_i}^{\overline{x}_i} \underline{U}^\omega(x)\,\mathrm{d}\omega; \text{ and } \overline{U}^j(x) := \frac{1}{\overline{y}_j - \underline{y}_j}\int_{\underline{y}_j}^{\overline{y}_j} \overline{U}^\omega(x)\,\mathrm{d}\omega,$$

for all $x \in \mathbb{R}$. Notice that, by construction, $\underline{U}^i, \overline{U}^j \in \mathcal{F}$ and $\mathbb{Q}^\tau(\underline{U}^i) = \{\overline{x}_i\}$, $\mathbb{Q}^\tau(\overline{U}^j) = \{\underline{y}_j\}$, for all $i \in I$ and $j \in J$. For any $\omega \in \text{supp}(H)$, let $F_\omega \in \mathcal{F}$ be defined as[1]

$$F_\omega(x) := \begin{cases} \underline{U}^\omega(x), & \text{if } \omega \in [0,\overline{x}]\setminus \cup_{i\in I} [\underline{x}_i, \overline{x}_i] \\ \underline{U}^i(x), & \text{if } \omega \in [\underline{x}_i, \overline{x}_i] \\ \overline{U}^\omega(x), & \text{if } \omega \in [\underline{y}, 1]\setminus \cup_{j\in J} [\underline{y}_j, \overline{y}_j] \\ \overline{U}^j(x), & \text{if } \omega \in [\underline{y}_j, \overline{y}_j] \end{cases},$$

for all $x \in \mathbb{R}$.

Now define $\tilde{\mu}$ as

$$\tilde{\mu}(\{F_\omega \in \mathcal{F} | \omega \leq x\}) := H(x),$$

for all $x \in \mathbb{R}$. By construction, $\text{supp}(\tilde{\mu}) = \{\underline{U}^\omega\}_{\omega\in[0,\overline{x}]\setminus\cup_{i\in I}[\underline{x}_i,\overline{x}_i]} \cup \{\underline{U}^i\}_{i\in I} \cup \{\overline{U}^\omega\}_{\omega\in[\underline{y},1]\setminus\cup_{j\in J}[\underline{y}_j,\overline{y}_j]} \cup \{\overline{U}^j\}_{j\in J}$. Furthermore, for any $x \in [0, 1]$,

$$\int_{\mathcal{F}} F(x)\tilde{\mu}(\mathrm{d}F) = \int_0^1 F_\omega(x)H(\mathrm{d}\omega) = x,$$

and hence $\tilde{\mu} \in \mathcal{M}^0_\tau(U)$. Then, for all $\omega \in \mathbb{R}$, $H^\tau(\omega|\tilde{\mu}) = H(\omega)$ for all $\omega \in \mathbb{R}$, as desired.

Now, consider any $H \in \mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon}$. Since $\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon}$ is a convex and compact set in a metrizable space, Choquet's theorem implies that there exists a probability measure $\Lambda_H \in \Delta(\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon})$ that assigns probability 1 to $\text{ext}(\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon})$ such that

$$H(\omega) = \int_{\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon}} \widetilde{H}(\omega)\Lambda_H(\mathrm{d}\widetilde{H}).$$

In the meantime, define the linear functional $\Xi : \mathcal{M}^0_\tau(U) \to \mathcal{F}$ as

$$\Xi(\tilde{\mu})[\omega] := \tilde{\mu}(\{F \in \mathcal{F} | F^{-1}(\tau) \leq \omega\}),$$

[1]Similar to the proof of Theorem 1, as a convention, define $\underline{U}^i(x) := \underline{U}^\omega(x)$ for all x if $\underline{x}_i = \overline{x}_i = \omega$. Similarly, define $\overline{U}^j(x) := \overline{U}^\omega(x)$ for all x if $\underline{y}_j = \overline{y}_j = \omega$.

for all $\tilde{\mu} \in \mathcal{M}^0_\tau(U)$ and for all $\omega \in \mathbb{R}$. Now define a probability measure $\widetilde{\Lambda}$ on $\mathcal{M}^0_\tau(U)$ by

$$\widetilde{\Lambda}_H(A) := \Lambda_H(\{\Xi(\tilde{\mu}) | \tilde{\mu} \in A\}),$$

for all $A \subseteq \mathcal{M}^0_\tau(U)$. Then, since $\Lambda(\mathrm{ext}(\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon})) = 1$ and since, for any $\widetilde{H} \in \mathrm{ext}(\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon})$, there exists $\tilde{\mu} \in \mathcal{M}^0_\tau(U)$ such that $H(\omega) = H^\tau(\omega|\tilde{\mu})$, it must be that $\widetilde{\Lambda}_H(\mathcal{M}^0_\tau(U)) = 1$, and hence $\widetilde{\Lambda}_H$ is a probability measure on $\mathcal{M}(U)$.

As a result, since Ξ is linear, let $\tilde{\mu} \in \mathcal{M}^0_\tau(U)$ be defined as

$$\tilde{\mu}(A) := \int_{\mathcal{M}^0_\tau(U)} \mu(A) \widetilde{\Lambda}_H(\mathrm{d}\mu),$$

for all measurable $A \subseteq \mathcal{F}$. Then, since Ξ is linear, it follows that

$$\begin{aligned} H(\omega) = \int_{\mathcal{I}^*_{\tau-\varepsilon,\tau+\varepsilon}} \widetilde{H}(\omega)\Lambda_H(\mathrm{d}\widetilde{H}) = & \int_{\mathcal{M}^0_\tau(U)} \Xi(\mu)[\omega]\widetilde{\Lambda}_H(\mathrm{d}\mu) \\ = & \Xi(\tilde{\mu})[\omega] \\ = & H^\tau(\omega|\tilde{\mu}), \end{aligned}$$

and therefore, $H \in \mathcal{H}^0_\tau$. By Lemma OA.1, it then follows that $H \in \mathcal{I}(\underline{F}_0^{\tau+\varepsilon}, \overline{F}_0^{\tau-\varepsilon})$ for all $\varepsilon > 0$ small enough. Thus, there exists $\bar{\varepsilon} > 0$ such that

$$\bigcup_{0<\varepsilon<\bar{\varepsilon}} \mathcal{I}(\underline{F}_0^{\tau+\varepsilon}, \overline{F}_0^{\tau-\varepsilon}) \subseteq \mathcal{H}^\tau_0 \subseteq \mathcal{I}(\underline{F}^\tau_0, \overline{F}^\tau_0).$$

This completes the proof. ■

OA.1.3 Exposed Points of $\mathcal{I}^*_\tau$

Lemma 2 in the main text characterizes the extreme points of $\mathcal{I}^*_\tau$, which are then used to construct a signal μ and a selection $r \in \mathcal{R}$ that generates an arbitrary $H \in \mathcal{I}^*_\tau$. In addition, the characterization of extreme points of $\mathcal{I}^*_\tau$ facilitates solving problems of the form

$$\sup_{H \in \mathcal{I}(\underline{F}^\tau_0, \overline{F}^\tau_0)} \int_{\mathbb{R}} V(\omega) H(\mathrm{d}\omega), \tag{OA.4}$$

for some measurable $V : \mathbb{R} \to \mathbb{R}$. A natural question is whether there are any irrelevant extreme points that never correspond to solutions of (OA.4). As we show in Theorem OA.1 below, the answer is "no."

Theorem OA.1. *Every extreme point of $\mathcal{I}^*_\tau$ is exposed.*

Proof. Consider any extreme point $\widehat{H}$ of $\mathcal{I}^*_\tau$. By Lemma 2, $\widehat{H}$ must take the form of (3). To show that H is an exposed point of $\mathcal{I}^*_\tau$, we construct a measurable function $V : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} V(\omega)\widehat{H}(\mathrm{d}\omega) \geq \int_{\mathbb{R}} V(\omega) H(\mathrm{d}\omega)$$

for all $H \in \mathcal{I}^*_\tau$, with the equality holding only at $H = \widehat{H}$. To this end, define $\underline{v}$ and $\overline{v}$ as

$$\underline{v}(\omega) := \begin{cases} -1, & \text{if } \omega \in [\underline{x}, \overline{x}) \backslash \cup_{i \in I} [\underline{x}_i, \overline{x}_i) \\ 0, & \text{otherwise} \end{cases}, \quad \text{and } \overline{v}(\omega) := \begin{cases} 1, & \text{if } \omega \in [\underline{y}, \overline{y}) \backslash \cup_{j \in J} [\underline{y}_j, \overline{y}_j) \\ 0, & \text{otherwise} \end{cases},$$

for all $\omega \in \mathbb{R}$, and let $V : \mathbb{R} \to \mathbb{R}$ be defined as

$$V(\omega) := \begin{cases} \int_{(-\infty,\omega)} [\underline{v}(\omega) + \overline{v}(\omega)]\widehat{H}(\mathrm{d}\omega), & \text{if } \omega < \overline{x} \\ \int_{(-\infty,\omega]} [\underline{v}(\omega) + \overline{v}(\omega)]\widehat{H}(\mathrm{d}\omega), & \text{if } \omega \geq \overline{x} \end{cases}.$$

Notice that V is upper-semicontinuous, strictly decreasing on $\mathrm{supp}(\widehat{H}) \cap [\underline{x}, \overline{x}]$, strictly increasing on $\mathrm{supp}(\widehat{H}) \cap [\underline{y}, \overline{y}]$, and is constant otherwise. Moreover, for any $i \in I$ and for any $\omega \leq \underline{x}_i$, $V(\omega) > V(\overline{x}_i)$; while for any $j \in J$ and for any $\omega \geq \overline{y}_j$, $V(\omega) > V(\underline{y}_j)$. Thus, for any $H \in \mathcal{I}^*_\tau$, it must be that

$$\int_{\mathbb{R}} V(\omega)H(\mathrm{d}\omega) \leq \int_{\mathbb{R}} V(\omega)\widehat{H}(\mathrm{d}\omega). \tag{OA.5}$$

Furthermore, by the definition of V, (OA.5) is binding only if $H = \widehat{H}$. Therefore, $\widehat{H}$ is exposed. ■

OA.2 Omitted Details

OA.2.1 Optimality of H^*

Here, we show that H^* defined in (5) is a solution of (4) when W is quasi-concave with a peak at $1/2$. Notice that, for any realized $x \in [0,1]$, and for any $H \in \mathcal{I}(\underline{F}^\tau_0, \overline{F}^\tau_0)$, if $x \leq F_0^{-1}(1/4)$, then $W(1 - H(x^-)) = W(H(x^-)) \leq W(\underline{F}_0^{1/2}(x^-))) = W(1 - \underline{F}^\tau_0(x))$, since $\underline{F}_0^{1/2}(x^-) \leq 1/2$ and W is increasing on $[0, 1/2]$; if $x \geq F_0^{-1}(3/4^+)$, then $W((1 - H(x^-))) = W(H(x^-)) \leq W(\overline{F}^\tau_0(x^-)) = W((1 - \overline{F}_0^{1/2}(x^-)))$, since $\overline{F}^{1/2}(x^-) \geq 1/2$ and W is decreasing on $[1/2, 1]$; while if $x \in (F_0^{-1}(1/2)(1/4), F_0^{-1}(3/4^+))$, clearly $W(1 - H(x^-)) \leq W(1/2)$. Therefore, for any $x \in [0,1]$, $W((1 - H(x^-))) \leq W(1 - H^*(x^-))$, which in turn implies that

$$\int_0^1 W((1 - H(x^-)))G(\mathrm{d}x) \leq \int_0^1 W((1 - H^*(x^-)))G(\mathrm{d}x).$$

Moreover, when W is strictly quasi-concave, and when both F_0 and G have full support on the same interval, the inequality must be strict for any $H \neq H^*$ with positive G-measure.

OA.2.2 Optimality of H^{**}

To see that H^{**} is optimal, note that for any $H \in \mathcal{I}(\underline{F}^\tau_0, \overline{F}^\tau_0)$,

$$\begin{aligned} \int_0^1 v_S(\omega)H(\mathrm{d}\omega) &= \int_0^{\omega_0} v_S(\omega)H(\mathrm{d}\omega) + \int_{\omega_0}^1 v_S(\omega)H(\mathrm{d}\omega) \\ &\leq v_S(\omega_0)H(\omega_0) + \int_{\omega_0}^1 v_S(\omega)H(\mathrm{d}\omega) \\ &\leq v_S(\omega_0)\underline{F}^\tau_0(\omega_0) + \int_{\omega_0}^1 v_S(\omega)\underline{F}^\tau_0(\mathrm{d}\omega) \\ &= \int_0^1 v_S(\omega)H^{**}(\omega), \end{aligned}$$

where the first inequality follows from $v_S(\omega) \leq v_S(\omega_0)$ for all $\omega \leq \omega_0$, and the second inequality follows from $H \preceq \underline{F}^\tau_0$ and v_S being nonincreasing on $[\omega_0, 1]$.

OA.2.3 Optimal Signal for a Quasi-Convex v_S

Consider the same problem in (8) with $\Omega = [0,1]$, and consider the case where v_S is quasi-convex with a minimum at $\omega_0 \in [0, F_0^{-1}(\tau)]$. For any $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$, let $\overline{H}$ be defined as

$$\overline{H}(\omega) := \begin{cases} \underline{F}_0^\tau(\omega), & \text{if } \omega < \underline{\omega}_H \\ \underline{F}_0^\tau(\underline{\omega}_H), & \text{if } \omega \in [\underline{\omega}_H, \overline{\omega}_H) \\ \overline{F}_0^\tau(\omega), & \text{if } \omega \geq \overline{\omega}_H \end{cases},$$

for all $\omega \in [0,1]$, where $\underline{\omega}_H$ and $\overline{\omega}_H$ are defined so that $\underline{F}_0^\tau(\underline{\omega}_H) = H(\omega_0) = \overline{F}_0^\tau(\overline{\omega}_H)$. Then, since v_S is nonincreasing on $[0, \omega_0]$ and is nondecreasing on $[\omega_0, 1]$,

$$\int_0^1 v_S(\omega) H(\mathrm{d}\omega) \leq \int_0^1 v_S(\omega) \overline{H}(\mathrm{d}\omega). \tag{OA.6}$$

Thus, it is without loss to search across $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$ that take the form of

$$H(\omega) := \begin{cases} \underline{F}_0^\tau(\omega), & \text{if } \omega < \underline{\omega}^\alpha \\ \underline{F}_0^\tau(\underline{\omega}^\alpha), & \text{if } \omega \in [\underline{\omega}^\alpha, \overline{\omega}^\alpha) \\ \overline{F}_0^\tau(\omega), & \text{if } \omega \geq \overline{\omega}^\alpha \end{cases}, \tag{OA.7}$$

for all $\omega \in [0,1]$, where $\underline{\omega}^\alpha$ and $\overline{\omega}^\alpha$ are defined so that $\underline{F}_0^\tau(\underline{\omega}^\alpha) = \alpha = \overline{F}_0^\tau(\overline{\omega}^\alpha)$. Hence, the solution of (8) must take the form of (OA.7) with $\alpha \in [0, \underline{F}_0^\tau(\omega_0)]$ that maximizes

$$\frac{1}{\tau}\int_0^{\underline{\omega}^\alpha} v_S(\omega) F_0(\mathrm{d}\omega) + \frac{1}{1-\tau}\int_{\overline{\omega}^\alpha}^1 v_S(\omega) F_0(\mathrm{d}\omega),$$

whose first-order condition, when F_0 has a density $f_0 > 0$, is simply $v_S(\underline{\omega}^\alpha) = v_S(\overline{\omega}^\alpha)$. In this case, the solution can be characterized by the system of equations

$$\underline{F}_0^\tau(\underline{\omega}) = \overline{F}_0^\tau(\overline{\omega}) \text{ and } v_S(\underline{\omega}) = v_S(\overline{\omega}). \tag{OA.8}$$

Moreover, if v_S is strictly quasi-convex and if F_0 has full support on $[0,1]$, then the inequality in (OA.6) must be strict, unless H takes the form of (OA.7). In this case, the solution is unique. Note that this solution cannot be attained by the "single-dipped" signal defined by Kolotilin, Corrao, and Wolitzky (2022), as noted in footnote 17.

As an example, suppose that $F_0 = U$. If $v_S(\omega) = (\omega - 1/2)^2$, then (OA.8) has a unique solution where $\underline{\omega} = 1/4$ and $\overline{\omega} = 3/4$, and hence H^* is optimal.

OA.2.4 Theorem 1 and the Securability Theorem

In section 5.1, we demonstrate how Theorem 1 can be applied to characterize the sender's equilibrium payoffs in a cheap talk game with transparent motives (as in Lipnowski and Ravid, 2020), where the receiver's optimal actions are τ-quantiles for each posterior. We prove our claims in the main text formally in this section. Recall that for any upper-semicontinuous function v_S and for any $v^* \in \mathbb{R}$, the set $\{\omega \in [0,1] | v_S(\omega) < v^*\}$, if non-empty, can be written as the union of countably many disjoint open intervals $\{(\underline{\omega}^i_{v^*}, \overline{\omega}^i_{v^*})\}_{i=1}^{I_{v^*}}$ for some $I_{v^*} \in \mathbb{N} \cup \{\infty\}$. Below we formally state the characterization of the sender's equilibrium payoffs.

Proposition OA.1. *For any $v^* \geq v_0 := \max_{\omega\in\mathbb{Q}^\tau(F_0)} v_S(\omega)$, there exists a perfect Bayesian equilibrium in which the sender's payoff is v^* if and only if $\overline{F}_0^\tau(\overline{\omega}_{v^*}^i) \leq \underline{F}_0^\tau(\underline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$.*

Proof. By theorem 1 of Lipnowski and Ravid (2020), it suffices to show that for any $v^* \geq v_0$, $\overline{F}_0^\tau(\overline{\omega}_{v^*}^i) \leq \underline{F}_0^\tau(\underline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$ if and only if there exists a signal $\mu \in \mathcal{M}$ and a receiver's best response function such that the sender's payoff under μ is at least v^* with probability 1. By Theorem 1, the latter is equivalent to the existence of some $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$ such that H assigns probability zero to the set $\{\omega \in [0,1] | v_S(\omega) < v^*\} = \cup_{i=1}^{I_{v^*}} (\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)$.

Consider any $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$ such that H is a constant on $(\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$. Then Corollary 2 implies that $\underline{F}_0^\tau(\underline{\omega}_{v^*}^i) \geq \overline{F}_0^\tau(\overline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$. Conversely, suppose that $\overline{F}_0^\tau(\overline{\omega}_{v^*}^i) \leq \underline{F}_0^\tau(\underline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$. First consider the case where $I_{v^*} < \infty$. In this case, it is without loss to reorder the intervals $\{(\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)\}_{i=1}^{I_{v^*}}$ so that $\overline{\omega}_{v^*}^i < \underline{\omega}_{v^*}^{i+1}$. Now let H be defined as

$$H(\omega) := \begin{cases} \underline{F}_0^\tau(\omega), & \text{if } \omega \notin \cup_{i=1}^{I_{v^*}} [\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i) \\ \underline{F}_0^\tau(\underline{\omega}_{v^*}^i), & \text{if } \omega \in [\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i) \end{cases},$$

for all $\omega \in \mathbb{R}$. Then, since $\overline{F}_0^\tau(\overline{\omega}_{v^*}^i) \leq \underline{F}_0^\tau(\underline{\omega}_{v^*}^i)$ for all $i \in \{1, \ldots, I_{v^*}\}$, it must be that $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$. Now consider the case where $I_{v^*} = \infty$. Since $\cup_{i=1}^\infty (\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i) \subseteq [0,1]$, it must be that $\sum_{i=1}^\infty (\overline{\omega}_{v^*}^i - \underline{\omega}_{v^*}^i) \leq 1 < \infty$. Thus, for any $\varepsilon > 0$, there exists $\bar{I}_{v^*}^\varepsilon \in \mathbb{N}$ such that $\sum_{i=\bar{I}_{v^*}^\varepsilon}^\infty (\overline{\omega}_{v^*}^i - \underline{\omega}_{v^*}^i) < \varepsilon$. If $\sum_{i=1}^\infty (\overline{\omega}_{v^*}^i - \underline{\omega}_{v^*}^i) = 1$, then $H(\omega) := \sum_{i=1}^\infty \mathbf{1}_{[\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)}(\omega) \underline{F}_0^\tau(\underline{\omega}_{v^*}^i)$ is well-defined on $[0,1]$ and is in $\mathcal{F}$. Moreover, $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$ by definition. In contrast, if $\sum_{i=1}^\infty (\overline{\omega}_{v^*}^i - \underline{\omega}_{v^*}^i) < 1$, then for $\varepsilon = (1 - \sum_{i=1}^\infty (\overline{\omega}_{v^*}^i - \underline{\omega}_{v^*}^i))/2$, take $\bar{I}_{v^*}^\varepsilon$. It then follows that there exists a finite collection of disjoint intervals $\{\underline{x}_{v^*}^j, \overline{x}_{v^*}^j\}_{i=j}^{\bar{I}_{v^*}^\varepsilon}$ such that, for all $i \in \mathbb{N}$, $(\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i) \subseteq (\underline{x}_{v^*}^j, \overline{x}_{v^*}^j)$ for some $j \in \{1, \ldots, \bar{I}_{v^*}^\varepsilon\}$ and $\underline{F}_0^\tau(\underline{x}_{v^*}^i) \geq \overline{F}_0^\tau(\overline{x}_{v^*}^i)$. Then, as shown above, there exists $H \in \mathcal{I}(\underline{F}_0^\tau, \overline{F}_0^\tau)$ such that H is constant on $(\underline{x}_{v^*}^j, \overline{x}_{v^*}^j)$ for all $j \in \{1, \ldots, \bar{I}_{v^*}^\varepsilon\}$ and hence constant on $(\underline{\omega}_{v^*}^i, \overline{\omega}_{v^*}^i)$ for all $i \in \mathbb{N}$, as desired. ∎

To take an example, we consider the case in the main text where $\tau = 1/2$, $F_0 = U$, and $v_S(\omega) = (\omega - 1/2)^2$. Notice that for any $v^* \geq v_0 = 0$, $I_{v^*} = 1$, and $\underline{\omega}_{v^*}^1 = 1/2 - \sqrt{v^*}$, and $\overline{\omega}_{v^*}^1 = 1/2 + \sqrt{v^*}$. Therefore, $\underline{F}_0^{1/2}(\underline{\omega}_{v^*}^1) \geq \overline{F}_0^{1/2}(\overline{\omega}_{v^*}^1)$ if and only if $2(1/2 - \sqrt{v^*}) \geq 2(1/2 + \sqrt{v^*}) - 1$, which simplifies to $v^* \leq 1/16$. As a result, v^* is the sender's equilibrium payoff under some perfect Bayesian equilibrium if and only if $v^* \in [0, 1/16]$.

Comparing the two models with and without sender commitment in this special case, we make note that the sender's payoff that can be induced by the optimal signal H^* with commitment is disjoint with the set of payoffs that are securable without commitment.

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